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NEW
HIGH SCHOOL ALGEBRA

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PART I

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PREFACE

This text contains sufficient material for the customary three semester courses in algebra, and will be found to meet adequately the varied college entrance requirements. The first part of the text is identical with the corresponding part of the authors' First Year Algebra. Attention is directed to the following features of the text:

By changing the traditional order of topics, some of the confusing difficulties of algebra are postponed until the third semester. Note the omission from Chapter VIII of certain types of factoring and of the generalized forms of even the simple types. These topics are gathered together in Chapter XVI, where they form a desirable review and extension of Chapter VIII. Obviously this chapter may be taken immediately after Chapter VIII if desired. Note also that Chapter XIV contains only so much of evolution and radicals as is required in the solution of quadratic equations. The subjects exponents and radicals are treated toward the end of the course. These chapters also may be taken in their traditional order, before quadratics, if desired. Note also that Chapter II contains only addition and multiplication of signed numbers, leaving subtraction and division until a later time. Note finally the Chapter XXVI containing supplementary topics. Many teachers will wish to use some of these topics in connection with earlier chapters.

In the early part of the text especially, each topic taken up is used in the solution of equations. (See §§ 9, 10, 12, 41, 51, 60, 107, etc.) In this manner the study of the topics is made purposeful and the equation receives desirable emphasis.

Problems are introduced at short intervals. Informational, geometric, and physics problems are used, as well as other valuable types. New types are introduced gradually, are taught with extreme care, appearing first in classified lists and thereafter in miscellaneous lists. Experimental verification is suggested for some of the facts from geometry and physics that are used. (See Exercises 7, 25, 28, 39, 49, 106; §§ 13, 142, 143, 190, etc.)

The abstract drill examples are simple rather than complex, are graded with extreme care, and are sufficient in quantity to meet the needs of the average class.

Mechanical processes like "transposition" and "clearing of fractions" are not introduced until the student is familiar with the underlying principles. In this manner thoughtful solution of exercises by the student is made habitual.

Efficiency in arithmetic is maintained and increased by the use of fractional and decimal coefficients, by requiring evaluation of expressions, and by expressing quadratic surds in their approximate decimal form. (See in this connection Chapters XIV, XV, and XXI.)

Formulæ are introduced as one of the most practical uses of algebra. (See §§ 17 and 146.) Other applications of algebra are found in §§ 44, 84, 143, 150, 190.

The data for informational problems are, in the main, of permanent rather than temporary interest, and of general rather than local interest.

Graphical representation and graphical methods are introduced from the secondary school point of view. They are viewed as a means of instruction rather than an end. The data for statistical graphs contain only two, or at most three, significant figures.

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ALGEBRA

INTRODUCTION

ALGEBRA is like arithmetic in some respects. Arithmetic consists of the study of addition, subtraction, multiplication, and division of some kinds of numbers, and of the application of this knowledge to some of the common problems of daily life and of business. Algebra continues this study of numbers. In arithmetic, numbers are represented by the digits 1, 2, 3, etc.; sometimes also, they are represented by letters, as, for example, in interest problems, where the principal is represented by P, the rate per cent by R, and the interest by I These letters make it possible to abbreviate rules; thus, the rule "the interest for one year equals the principal multiplied by the rate per cent," may be expressed by the letters as follows:

$$I = P \times \frac{R}{100}.$$

In algebra, letters are regularly employed to represent numbers. Some new kinds of numbers and many new mathematical ideas are studied, and, as in arithmetic, some of the uses of this knowledge are illustrated.

Algebra has a very long history. A little was known about it centuries before the Christian Era. The oldest mathematical book which we have, written by an Egyptian named Ahmes, contains some problems similar to those found in our algebras. Ahmes lived before 1700 B.C. Knowledge of algebra grew very slowly indeed for many centuries; in fact it was not until the sixteenth century that algebra assumed the form which it has to-day, and since then many discoveries and improvements in it have been made. Many of the wisest mathematicians of former days contributed to this growth. Thanks to their combined achievements and ingenuity, it is now possible for any boy or girl in the first year of high school to get a much broader view of the elementary part of the subject than many of these men had.

Scattered through the text, will be found historical notes calling attention to some of the epoch-making innovations in the development of algebra, together with the name and time of the man making the step forward.

I. LITERAL NUMBER

1. In arithmetic, numbers are represented by the *digits* 1, 2, 3, 4, 5, 6, 7, 8, 9, and 0, and combinations of them In **Algebra,** numbers are also represented by letters. Numbers represented by letters are called **Literal Numbers**. The following examples illustrate the use of letters as numbers.

Example 1. If a boy saves 5 cents per day, how much does he save:

(*a*) in 3 days? (*b*) in 5 days?

(*c*) in any number of days?

This last result may be expressed by saying, "as many cents as are obtained by finding the product of the number of days and 5."

In algebra, it may be expressed thus:

Let $n =$ the number of days.

Then, $5 \times n =$ the number of cents saved.

So, if n is 6, $5 \times n$ is 5×6 or 30;

if n is 8, $5 \times n$ is 5×8 or 40.

Example 2 How many inches are there:

(*a*) in 9 feet? (*b*) in any number of feet?

Let $x =$ the number of feet.

Then, $12\,x =$ the number of inches in x feet.

$12\,x$ is read "twelve x."

2. Sign of Multiplication. The symbol, ×, is used to indicate multiplication in algebra as well as in arithmetic; it is read "*times*" or "*multiplied by*." A dot, $\cdot$, placed above the line, is also used as a sign of multiplication, and generally even the dot is omitted, so that $12 \times m$ may be written $12 \cdot m$ or $12\,m$. $a \times b$ may be written $a \cdot b$ or ab, and is read "*a b*."

Historical Note. — The symbol, ×, was first used by an Englishman, Oughtred, about 1631. The symbol, $\cdot$, was introduced by Leibnitz in 1693. Multiplication was indicated as early as the thirteenth century, in Hindu and Italian books, by simply writing the factors side by side. This method was forgotten for a time, and was reintroduced by German algebraists during the fifteenth century.

3. The result obtained by multiplying two or more numbers together is called the **Product**, and the numbers are called the **Factors** of the product.

EXERCISE 1

1. What does $10\,d$ mean? $5\,r$? $6\,s$?

2. How much is $10\,d$, when d is 2? 3? 5? 6?

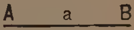
3. How much is $7\,r$, when r is 4? 6? 12? $\frac{5}{7}$?

Another way of expressing this example is to say: "what is the value of $7\,r$ when r is 4?"

4. Find the value of $8\,a$ when a is 5; 15; 2.5; $\frac{3}{4}$.

5. Find the value of $9\,W$ when W is 8; 12; $\frac{4}{3}$.

6. If a equals the number of inches in the line AB, what does $3\,a$ equal? Illustrate it.



7. If b represents the number of square feet in a rectangle, what does $2\,b$ represent? $\frac{2}{3}\,b$? $6\,b$?

8. If a man earns \$3 per day, what will he earn in 6 days? in 20 days? in n days?

9. If a book costs 75¢, what will three of them cost? x of them? How much is $75x$¢, when x is 4?

10. If a train travels at the rate of 25 miles per hour, how far will it go in 3 hours? in 5 hours? in x hours?

11. One cubic foot of water weighs 62.5 pounds. How much do x cubic feet weigh? How many pounds are $62.5x$ pounds when x is 4? 5? 10?

12. If a farm consists of 85 acres, valued at A dollars per acre, what is the value of the farm? What is it when A is 75?

13. If a man receives y dollars per week, how much will he receive in a year? Find the amount if y is 22.

14. If each of 35 persons contributes s dollars to the expense of an excursion, what is the total expense? Find it when s is 5; 6.

15. If n represents a number, what will represent a number 3 times as large? 5 times? $2\frac{1}{2}$ times? Find the value of each of these when n is 2; 6; 10.

4. The symbols ( ) are called *parentheses*. In mathematics, they mean that the numbers within are to be combined as the signs indicate, and that the result is to be treated as a whole.

Thus, $(3 \times 5) - (7 + 3)$ means: multiply 3 by 5; add 7 and 3; subtract the second result from the first.

5. An Important Multiplication Law. When several numbers are to be multiplied together, the product may be found by multiplying the first factor by the second, that result by the third, and so on.

Example 1. $$\begin{aligned} 2 \times 3 \times 4 \times 5 &= (2 \times 3) \times 4 \times 5 = 6 \times 4 \times 5 \\ &= (6 \times 4) \times 5 = 24 \times 5 = 120. \end{aligned}$$

Example 2. If a square contains $4a$ square feet, find the area of a square 5 times as large.

Solution: 1. The area is $5 \times 4a$ square feet or $(5 \times 4) \times a = 20 \times a = 20a$ sq. ft.

2. This result is true for any value of a;

if $a = 3,\ 4a = 12,$

and $5 \times 4a = 5 \times 12 = 60;$

also $20a = 20 \times 3 = 60.$

The fact that the result is 60 in both cases shows that the solution is probably correct.

EXERCISE 2

Find, as in Examples 1 and 2, the results in the following examples, and test the results as in 2 for some particular value of the literal number:

1. $6 \times 8a.$

2. $7 \times 10b.$

3. $9 \times 8n.$

4. $9 \times 17z.$

5. $3 \times \frac{1}{6}m.$

6. $5 \times \frac{3}{5}k.$

7. $8 \times \frac{7}{8}t.$

8. $12 \times \frac{2}{3}x.$

9. If one number is represented by $2b$, what will represent a number 3 times as large? one third as large?

10. If John is 4 times as old as James, and if James is $2y$ years of age, how old is John? Find both ages if y is 3.

11. If the volume of a sphere is $163t$ cubic inches, what is the volume of a sphere 3 times as large?

12. If the interest on a sum of a money is $25r$ dollars for one year, what is the interest for 4 years? 3 years? 6 years?

13. There are three numbers of which the first is 4 times the second, and the third is 3 times the first. Represent the second number by s, and find the others. Find their values when the number s is 5.

14. There are three numbers of which the second is 8 times the first, and the third is 4 times the second. Let f represent the first, and then represent the others.

15. The value of A's property is 5 times that of B's, and the value of C's property is 4 times that of A's. Represent the number of dollars B possesses by b, and then represent the number of dollars owned by A and C.

6. An Important Division Law.

Since $2 \times 3a = 6a$, then $6a \div 2 = 3a$.
Similarly, $40x \div 5 = 8x$, since $5 \cdot 8x = 40x$.

Rule. — To divide the product of an arithmetical number and a literal number by an arithmetical number:

1. **Find the quotient of the arithmetical numbers.**
2. **Multiply the quotient of step 1 by the literal number.**

EXERCISE 3

1. Divide each of the following numbers by 5:
   (*a*) $25t$. (*b*) $30x$. (*c*) $45rs$. (*d*) $75y$.

2. Divide each of the following numbers by 3:
   (*a*) $6r$. (*b*) $30c$. (*c*) $42d$. (*d*) $54e$.

3. Divide each of the numbers in Example 2 by 2.

4. What part of $36w$ is:
   (*a*) $3w$? (*b*) $4w$? (*c*) $6w$? (*d*) $1w$?

5. What part of $44x$ is:
   (*a*) $11x$? (*b*) $4x$? (*c*) $22x$? (*d*) $1x$?

Find the following quotients:

6. $39y \div 3$.
7. $96f \div 12$.
8. $81x \div 9$.
9. $49b \div 49$.
10. $120t \div 120$.
11. $45r \div 9$.
12. $63s \div 63$.
13. $72t \div 72$.
14. $25x \div 25$.

Historical Note. — The symbol, ÷, was introduced by John Pell who lived during the seventeenth century.

7. Use of Literal Numbers in Solving Problems.

Literal numbers aid in solving certain kinds of problems.

Example. How long will it take a bricklayer to lay 38,500 bricks if he can lay 3500 in one day?

ARITHMETICAL SOLUTION

Since he can lay 3500 bricks in one day, then in the unknown number of days he can lay 3500 times that number of bricks. Since this must be 38,500, according to the statement of the problem, then the number of days must be $\frac{1}{3500}$ of 38,500 or 11.

ALGEBRAIC SOLUTION

Let $n =$ the unknown number of days.
Then, $3500\,n =$ the number of bricks laid in these days,
and $38,500 =$ the number of bricks to be laid.
So, $3500\,n = 38,500.$

Since one n is $\frac{1}{3500}$ of $3500\,n$, divide these two equal numbers by 3500.

Then, $n = 11.$

TEST: 11 is correct, for $3500 \times 11 = 38,500$.

8. The mathematical statement $3500\,n = 38,500$ is called an **Equation.** The literal number in the equation is called the **Unknown Number.**

An **Equation** expresses the equality of two numbers.

The numbers on the right of the equality sign form the **Right Member** of the equation, and the ones on the left, the **Left Member.**

An equation implies a question: "for what value of the unknown number is the equality true?"

For example, in the equation of § 7, n can have only one value, — the one found, 11; thus, n cannot be 10, for 3500×10 is 35,000, and not 38,500.

Finding the value of the unknown is called **Solving the Equation.**

HISTORICAL NOTE. — The equation is implied in Ahmes' book. To indicate the unknown number, he used a word *hau* corresponding to our word *heap*. Diophantus, a Greek mathematician of the fourth century, used for the unknown the last letter, *s*, of the word for number; Vieta, a French mathematician of the sixteenth century, used the vowels, *A*, *E*, *I*, *O*, *U* and *Y*; Harriot, an English mathematician of about the same time, also used the vowels but wrote them with small letters; Descartes, a French mathematician of the same period, used the last letters of the alphabet, x, y, and z.

9. In solving the equation in § 7, two equal numbers were divided by the same number. It is clear that if equal numbers are divided by equal numbers, the quotients are equal.

This fact is used in algebra in the following form:

Rule. — Both members of an equation may be divided by the same number without destroying the equality.

EXAMPLE. Solve the equation:

$$36\,k = 468.$$

SOLUTION: 1. Since k is $\frac{1}{36}$ of $36\,k$, divide both members of the equation by 36.

2. $k = \frac{468}{36}$ (Rule § 9)
3. $= 13.$

EXERCISE 4

Solve the following equations:

1. $7\,p = 238.$
2. $8\,n = 608.$
3. $9\,x = 423.$
4. $6\,A = 312.$
5. $15\,x = 240.$
6. $27\,y = 351.$
7. $2\,x = 161.$
8. $5\,v = 218.$
9. $8\,m = 1864.$
10. $10\,w = 2345.$

The arithmetical solution of the following examples is easy. Their algebraic solution leads to the simplest form of equation. Give the algebraic solution.

11. What number multiplied by 13 equals 221?

12. The product of a certain number and 17 equals 408; find the number

13. A farm consisting of 43 acres is offered for sale at the price $3655. What is the average price per acre?

14. What number multiplied by 3.7 equals 8.51?

15. If the total expense for a picnic for a party of 18 boys and girls is $5.94, how much must each one contribute?

16 A man is compelled to make a journey of 126 miles in his automobile over a poor road in 7 hours; how many miles must he average per hour?

17. The fastest train on the Pennsylvania Railroad between St. Louis and New York makes the trip in 24 hours; if the distance is 1052.4 miles, what is the average rate per hour?

10. A second rule used in solving equations is:

Rule. — Both members of an equation may be multiplied by the same number without destroying the equality.

This fact may be illustrated by the scales Suppose that the sugar S balances the weight W; if the weight is doubled, then the weight of sugar must also be doubled in order to keep the balance.

Example 1. The circumference of one of the large redwood trees of California is 70 feet. Find its diameter (The circumference of a circle is twenty-two sevenths of its diameter.)

Solution: 1. Let $d =$ the number of feet in the diameter.

2. Then, $\frac{22}{7}d = 70$, the number of feet in the circumference.

3. Multiply both members of the equation by 7.

Then, $\overset{1}{\cancel{7}} \cdot \frac{22}{\cancel{7}} d = 7 \cdot 70,$ (§ 10)

or $22\,d = 490.$

4. Divide both members of the equation by 22.

Then, $d = \frac{490}{22} = 22\frac{3}{11}$, or 22.2+ feet.

Check: Does $\frac{22}{7}$ of $22\frac{3}{11} = 70$?

$$\frac{22}{7} \times 22\tfrac{3}{11} = \frac{\overset{2}{\cancel{22}}}{\cancel{7}} \times \frac{\overset{35}{\cancel{245}}}{\cancel{11}} = 70.$$

Example 2. Solve the equation $\frac{5}{8}x = 142$.

Solution. 1. $\frac{5}{8}x = 142$

2. Multiply both members of the equation by 8.

Then, $\overset{1}{\cancel{8}} \cdot \frac{5}{\cancel{8}}x = 8 \cdot 142,$ (Rule, § 10)

or $5x = 1136.$

3. Divide both members of the equation of step 2 by 5.

Then, $x = \frac{1136}{5} = 227.2.$ (Rule, § 9)

Check: Does $\frac{5}{\cancel{8}} \times \overset{28.4}{\cancel{227.2}} = 5 \times 28.4 = 142.0$? Yes.

EXERCISE 5

Solve the following equations and problems:

1. $8a = 280$
2. $15y = 345$
3. $27c = 1242.$
4. $76m = 1444$
5. $27.5x = 277.75$
6. $\frac{3}{5}x = 81.$
7. $\frac{4}{3}y = 188.$
8. $\frac{3}{7}z = 96.$
9. $\frac{11}{4}t = 429.$
10. $\frac{5}{9}r = 200$

11. Three tenths of the cost of a certain automobile is $210. Find the cost of the automobile.

12. The selling price of a certain book is $\frac{5}{4}$ of its cost. Find its cost if it sells for $1.50.

13. Five eighths of a certain number is 95. Find the number.

14. Thirteen ninths of a certain number is 143. Find the number.

15. Two fifths of the area of Lake Michigan is 9200 square miles. Find the area of Lake Michigan.

16. Three eighths of the cost of the Suez Canal was $37,500,000. Find the cost of the canal.

17. Seven twenty-fifths of the distance from New York to San Francisco is 910 miles. Find the distance from New York to San Francisco.

18. Many metal articles, like a brass candlestick, are made by pouring melted metal into a mold. The piece taken from the mold is called a casting.

In making a brass casting, $\frac{3}{200}$ of the metal is lost in the melting. How much brass must be melted to make a casting which will weigh 72 pounds? (Find the second decimal.)

19. Cottonseed meal is used as a fertilizer on farms. It contains about 7% of nitrogen, a necessary plant food. How many pounds of cottonseed meal must a farmer purchase who wishes to distribute 15 pounds of nitrogen over an acre of ground?

20. Tobacco stems also are used as a fertilizer. They contain about 8% of potash, another necessary plant food. How many pounds of tobacco stems must a farmer purchase who wishes to obtain 12 pounds of potash?

11. Addition and Subtraction of Numbers having a Common Factor.

A number which is a factor of two or more numbers is called a **Common Factor** of these numbers.

Thus, 3 is a common factor of 6 and 9.
a is a common factor of $4a$ and $7a$.
5 is a common factor of 3×5 and 2×5.

A short method of adding numbers which have a common factor is illustrated in the following examples.

Historical Note. — The symbol, +, was first used in print by a German mathematician, Widmann, in 1489. The origin of the symbol is much in doubt. Italian writers of this period used the symbol *p̃*, the first letter of the Latin word *plus*. One explanation given for the sign, +, is that it comes from an inverted *t*, ʇ. The Latin word, *et*, means *and*, and in place of it this inverted *t* was often used. It is easy to see how the symbol + may have been derived from the symbol, ʇ.

EXERCISE 6

1. 3 times 7 plus 2 times 7 is 5 times 7.

$$3 \times 7 \quad + \quad 2 \times 7 \quad = \quad 5 \times 7$$

for $$21 \quad + \quad 14 \quad = \quad 35.$$

2. $(12 \times 9)+(8 \times 9)=20 \times 9=?$

3. $(8 \times 4)+(7 \times 4)=(?)\times 4=?$

4 $(5 \times 7)+(6 \times 7)+(9 \times 7)=(?)\times 7=?$

5. 6 times n + 4 times n = (?) times n?

6. $6x+4x=(?)\,x?$

7. $7a+3a=?$

8. $11y+8y+5y=?$

9. $12v+6v+3v=?$

10. $2r+3r+5r+10r=?$

11. $(5 \times 4)-(2 \times 4)=3 \times 4=12$, for $20-8=12$.

12. $(10 \times 7)-(4 \times 7)=(?)\times 7=?$

13. $(12 \times 8)-(5 \times 8)=(?)\times 8=?$

14. $9x-5x=(?)\,x?$

15 $16b-5b=?$

16. $20y-10y=?$

17 $4m+6m-2m=?$

18. $13t-5t+9t-4t=?$

19. $12A+6A-2A+5A=?$

20. $8r+13r-11r+5r=?$

21. One number is four times another. Represent the smaller by s. Then represent the larger and find their sum.

22. One number is $\frac{2}{3}$ as great as another. Let b equal the larger. Represent the smaller and find their sum.

23. One number is 5 times as large as another. Let s equal the smaller. Represent the larger and then find their difference.

24. The base of a rectangle is three times the altitude. Represent the altitude by a; then represent the base. Find also the perimeter.

(The perimeter is the sum of the lengths of the sides.)

25. What is the perimeter of a triangle if one of its sides is $2t$ inches, if the second side is three times as long as the first and the third side is $2\frac{1}{2}$ times as long as the first?

26. How many inches in n feet? in n yards? in n yards + n feet + n inches?

27. How many cents in x nickels? in x dimes? in x dimes + x nickels + x cents?

Simplify the following:

28. $2c + \frac{1}{2}c$.
29. $3c + \frac{2}{3}c$.
30. $5x + \frac{2}{7}x$.
31. $3c + .2c$.
32. $2b + .25b$.
33. $4m + .15m$.
34. $\frac{1}{2}x + \frac{1}{6}x$.
35. $\frac{2}{3}y + \frac{1}{2}y$.

12. Addition of Literal Numbers used in Equations.

EXAMPLE 1. The sum of two numbers is 91. The greater number is 12 times the smaller. Find the numbers.

SOLUTION: 1. Let $s =$ the smaller number.
2. Then, $12s =$ the larger number.
3. Then, $s + 12s = 91$, since the sum of the numbers is 91.
4. Adding, $13s = 91$.
5. Dividing, $s = 7$.

CHECK: If the smaller number is 7, the larger must be 84 and their sum is 91.

EXERCISE 7

Solve and check the following equations:

1. $3a + 4a = 42$.
2. $4m + 5m = 108$.
3. $7b - b = 66$.
4. $3x + 7x = 120$.
5. $11y - 2y = 81$.
6. $6z + 5z = 99$.
7. $3x + 11x + 12x = 130$.
8. $15t + 8t - 3t = 20$.
9. $7r - 5r + 6r = 4$.
10. $18w - 7w + 9w = 65$.
11. $22x + 13x - 6x = 116$.
12. $16y - 3y + 4y = 102$.

13. The greater of two numbers is four times the smaller The sum of the numbers is 60. Find the numbers.

14. If five times a certain number is increased by three times the same number, the result is 168. Find the number.

15. Divide \$56 between A and B so that A shall receive seven times as much as B.

16. A, B, and C together have \$96. B has twice as much as C, and A has as much as B and C together. How much has each?

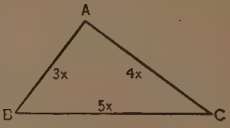
17. A man had \$4195. After spending a certain sum, he found that he had left four times as much as he had spent. How much did he spend?

18. The sum of three numbers is 120. The second is five times the first, and the third is nine times the first. Find the numbers.

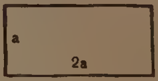
19. The sum of three numbers is 360. The second is fourteen times the first, and the third is the sum of the other two. Find the numbers.

20. Three men are asked to contribute to a fund. The first agrees to give twice as much as the second, and the third to give twice as much as the first. How much must each contribute to make a total of \$525?

21. The perimeter of the triangle *ABC* is 240 inches. Find the lengths of its sides.

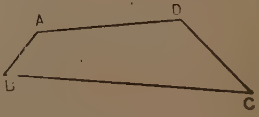


22. The perimeter of a rectangle is 132 inches. The base is double the altitude. Find the dimensions of the rectangle.



23. The length of the fence about a rectangular field is 320 rods. If the long dimension is three times the short dimension, find the length of each.

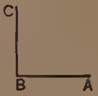
24. The perimeter of the quadrilateral *ABCD* is 220 inches. The side *CD* is twice as long as the side *AB*; the side *AD* is three times as long; the side *BC* equals the sum of the sides *AD* and *CD*. Find the length of each side.



25. The shortest distance by railroad from New York to Chicago is 10 times the distance from New York to Philadelphia. The sum of the two distances is 990 miles. Find the distance from New York to Chicago and to Philadelphia.

PROBLEMS ABOUT ANGLES

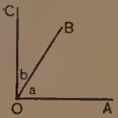
13. When two lines meet they form an **Angle** ($\angle$). The angle ABC is a **Right Angle.**



Angles are measured by a *unit* called a **Degree** (°).

A right angle contains 90°.

Two angles whose sum is a right angle are **Complementary Angles**; each of the angles is called the **Complement** of the other. The angles AOB and BOC are complementary; hence $a + b = 90$.



EXERCISE 8

1. How many degrees are there in one half of a right angle? one third?

2. What is the complement of 30°? 40°? 70°? a°? x°?

3. Are angles of 25° and 55° complementary? Why?

4. If the angles $3x$ and $7x$ are complementary, what is their sum? Form an equation and determine x. What are the angles?

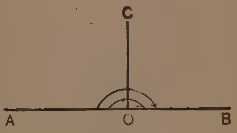
5. Determine the angles $5a$ and $4a$ if they are complementary.

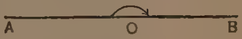
6. What angle is double its complement? (Let c equal the number of degrees in the complement; form an equation.)

7. What angle is three times its complement?

8. What angle is five times its complement?

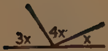
A **Straight Angle** equals two right angles.





What kind of angles are the angles AOC and BOC? How many degrees in their sum? How many degrees in $\angle AOB$? How many degrees in a straight angle?

9. Find each angle in the adjoining figure.



10. There are three angles whose sum is 180°. The second is double the first, and the third is the sum of the other two. Draw a figure to illustrate this problem. Find the angles.

Two angles whose sum is a straight angle are called **Supplementary Angles**; each of them is called the **Supplement** of the other.

11. What is the supplement of 50°? 90°? 100°? x°? $2a$°?

12. The angles $5x$ and $7x$ are supplementary How many degrees are there in each?

13. Find the angle which is four times its supplement.

14. Find the angle which is five times its supplement.

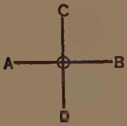


Fig. 1

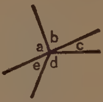


Fig. 2

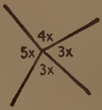


Fig. 3

The sum of all the angles around a point is 4 right angles or 360°.

Thus $a+b+c+d+e=360$.

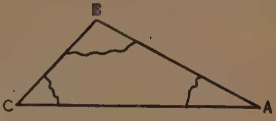
15 Find each of the angles in Figure 3.

16. There are four angles whose sum is the total angle around a point. The first angle contains a°; each of the others is double the preceding. Draw a figure to illustrate this problem. How many degrees are there in each angle?

17. There are four angles whose sum is the total angle around a point. The second angle is one half of the first; the third angle is three halves of the first; and the fourth angle is four times the third. Find the angles.

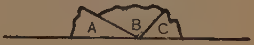
12 7
94

If the angles of triangle ABC are torn off and placed side by side, their sum is found to be 180°.



Draw a triangle with a ruler, and see if you find this to be true.

From this we conclude that the sum of the angles of a triangle is 180°.



18. The second angle of a triangle is double the first, and the third angle is six times the first. How many degrees are there in each?

19. Find the angles of a triangle when two of the angles are equal and the third is equal to the sum of the other two.

20. Find the angles of a triangle if the first is 4 times the second, and the third is 7 times the second.

DEFINITIONS

14. An **Algebraic Expression**, or simply an **Expression**, is a number expressed in algebraic symbols; as,

$$2,\ ab,\ 2x - 3zy,\ \frac{r}{s}.$$

The **Numerical Value** of an expression is found by substituting particular values for the literal numbers, and performing the *indicated* operations.

ab *indicates* that a is to be multiplied by b.

$\frac{r}{s}$ *indicates* that r is to be divided by s.

$2x - 3yz$ *indicates* that 3 times the product of y and z is to be subtracted from 2 times x.

15. If the same number is used as a factor one or more times to form a product, the result is called a **Power** of the number.

The number itself is called the **Base**.

An integer written at the right of and above the base, to indicate the number of times the base is used as a factor, is called an **Exponent**. Thus,

a^2, read "*a square*" or "*a second power*," means $a \times a$,
a^3, read "*a cube*" or "*a third power*," means $a \times a \times a$;
a^4, read "*a fourth*" or "*a fourth power*" means $a \times a \times a \times a$.
If no exponent is written, the exponent 1 is understood.

HISTORICAL NOTE. — Mathematicians sought suitable symbols for the powers of a number for a long time. At first words were used for them. Our "*a square*" and "*a cube*" owe their introduction to Greek mathematicians who called the second power by a word which means the square, and the third power by one which means the cube. Herigone, a French mathematician of the early part of the seventeenth century, wrote $a2$, $a3$, $a4$, etc., and finally Descartes, in 1637, introduced the present symbols. The word, *power*, comes from a Latin word, *potentia*, which corresponds to the Greek word used for the second power. The word, *exponent*, was introduced by Stifel about 1553.

16. The **Fundamental Operations** are addition, subtraction, multiplication, and division. Indicated operations are to be performed in the following order: first, all multiplications and divisions in their order from left to right; then all additions and subtractions from left to right.

EXAMPLE. Find the numerical value of the expression,

$$4\,ab + \frac{6\,c}{b} - d^3,$$

when $$a = 4,\quad b = 3,\quad c = 5,\quad d = 2.$$

SOLUTION: Substituting,

$$\begin{aligned} 4\,ab + \frac{6\,c}{b} - d^3 &= 4 \cdot 4 \cdot 3 + \frac{6 \cdot 5}{3} - 2^3 = 4 \cdot 4 \cdot 3 + \frac{6 \cdot 5}{3} - 2 \cdot 2 \cdot 2 \\ &= 48 + 10 - 8 = 50. \end{aligned}$$

EXERCISE 9

Find the numerical value of the following expressions when

$$a = 2,\quad b = 5,\quad c = 3,\quad d = 4,\quad m = 4,\quad n = 3.$$

1. $3\,a + 5\,b$.
2. $4\,c - 2\,d$.
3. $5\,m + 3\,n$.
4. $a^2 + c^2$.
5. $b^2 - d^2$.
6. $a^3 + n^3$.

7. $2c^2 - 3b$.

8. $\frac{2d}{m} + \frac{3c}{n}$.

9. $3ab + 2c - 4d$.

10. $5b + 6m - n^2$.

11. $\frac{a}{b} + \frac{c}{d} + \frac{n}{m}$.

12. $2a^2 - 3a + 1$.

13. $m^2 - mn + n^2$.

14. $a^3 + 3a^2b + 3ab^2$.

15. $\frac{2a + 4c - 3d}{m + n}$.

16. $c^a + b^a$.

Write in symbols the following and find their value:

17. The sum of a and b; c and d; m and n.

18. The difference between a and b; c and d.

(The difference between a and b is $b - a$.)

19. The product of a and b; c and d; m and n.

20. The quotient of a and b; c and d; m and n.

21. a increased by $2b$; c increased by $3d$.

22. The square of m increased by the square of n.

23 The cube of b decreased by the cube of m.

24. 10 more than $3a$; 5 less than $4b$.

25 3 more than the quotient of m divided by d.

26. 4 less than the product of a and c.

27. The sum of the squares of a and b.

FORMULÆ

17. When a rule of computation is expressed by means of algebraic symbols, the result is called a **Formula**.

EXAMPLE 1. Find the formula for the area of a rectangle.

SOLUTION: The area equals the product of the base and altitude.

Let a = the number of units in the altitude.

Let b = the number of units in the base.

Let A = the number of square units in the area;

then,

$$A = ab.$$

The following examples show how to use a formula.

Example 2. Find the area of a rectangle whose altitude is 8 inches and whose base is 15 inches.

Solution: 1. Use the formula $A = ab$.
2. Substitute 8 for a, and 15 for b.
Then, $A = 8 \times 15$ or 120.

Example 3. Find the altitude of a rectangle whose base is 75 feet and whose area is 675 square feet.

Solution: 1. Substitute in the formula $A = ab$.
$A = 675;\ b = 75$.
2. Then, $675 = a \times 75$,
or $675 = 75\,a$.
3. Divide both members by 75: $9 = a$.

Example 4. Find the base of a rectangle whose altitude is 11 inches and whose area is 385 square inches.

Solution: 1. Substitute in the formula $A = ab$.
2. $A = 385;\ a = 11$.
Then, $385 = 11 \times b$.
3. Divide both members by 11: $\frac{385}{11} = b$,
or $b = 35$.

Rule. — To solve a problem by a formula:

1. For the known letters in the formula substitute their values.

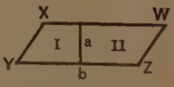
2. Perform all of the indicated operations.

3. If an equation is formed, solve for the unknown letter, if possible.

EXERCISE 10

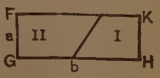
1. The figure $XYZW$ is a **Parallelogram** (▱). Find a formula for determining its area.

(*a*) Let b and a equal the units in the base and the altitude, and A the square units in the area.



(*b*) How does $XYZW$ compare in area with $FGHK$?

(*c*) What is $FGHK$? What is its base? altitude? area?



(*d*) What then is the area of $XYZW$?

(*e*) Make a rule for finding the area of a parallelogram.

(*f*) Expressed as a formula, this rule is

$$A = a \cdot b$$

(*g*) Find A when $a = 12$ and $b = 20$.

(*h*) Find A when $a = 15$ and $b = 25$.

(*i*) Find a when $A = 500$ and $b = 40$.

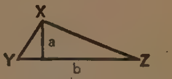
(*j*) Find a when $A = 600$ and $b = 25$.

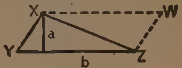
(*k*) Find b when $A = 750$ and $a = 15$.

(*l*) Find b when $A = 960$ and $a = 32$.

2. The figure XYZ is a **Triangle** ($\triangle XYZ$).

Find a formula for determining its area.





(*a*) Let b, a, and A represent the units in the base, altitude, and area respectively.

(*b*) What is the figure $XYZW$? what is its base? altitude? area?

(*c*) What part of ▱ $XYZW$ is $\triangle XYZ$?

(*d*) What then is the area of $\triangle XYZ$?

(*e*) Make a rule for finding the area of a triangle.

(*f*) Expressed as a formula, this rule is,

$$A = \frac{1}{2}ab.$$

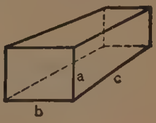
(*g*) Find A when $a = 10$ and $b = 17$.

(*h*) Find A when $a = 20$ and $b = 30$.

(*i*) Find b when $A = 260$ and $a = 40$.

(*j*) Find a when $A = 1200$ and $b = 60$.

3. The figure of the rectangular solid has the dimensions indicated.



Find a formula for determining its volume.

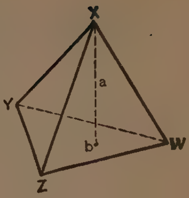
Let V represent the number of units in the volume. The volume equals the product of the three dimensions.

(*a*) Express this rule as a formula.

(*b*) Find V when $a = 3$, $b = 5$, $c = 8$.

(*c*) Find V when $a = 6$, $b = 9$, $c = 7$.

(*d*) Find c when $V = 240$, $a = 6$, and $b = 5$.



4. The figure $XYZW$ is a **Pyramid.**

Its volume equals one third of the product of its base and altitude.

(*a*) Express this rule as a formula, letting V equal the number of units in the volume.

(*b*) Find V when $a = 18$, $b = 15$.

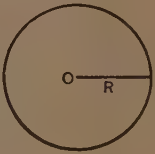
(*c*) Find b when $V = 160$, $a = 24$.

(*d*) Find a when $V = 900$, $b = 30$.

to here problem

5. The formula for the circumference of a circle is:

$$C = 2\pi R,$$



where $C =$ the number of units in the circumference,
where $R =$ the number of units in the radius,
where $\pi = 3.1416$ (π is read "pi").

(*a*) Express this rule in words.

(*b*) Find, by the formula, C when R is 10 inches.

(*c*) Find, by the formula, R when C is 628.32 inches.

6. The formula for the area of a circle is:

$$A = \pi R^2.$$

(*a*) Express this rule in words.

(*b*) Find, by the formula, A when R is 10 inches.

(*c*) Find, by the formula, A when R is 5 feet.

7. The numbers s, v, t, and g are connected by the formula

$$s = vt + \frac{1}{2}gt^2;$$

find s when $v = 50$, $t = 3$, $g = 32.16$.

8. From the formula $E = \frac{wv^2}{2g}$, find E when $w = 75$, $v = 50$, $g = 32.16$. (Carry the result to one decimal place.)

9. From the formula $V = \frac{4}{3}\pi R^3$, find V when R is 3.

10. From the formula $S = 4\pi R^2$, find S when $R = 5$.

II. POSITIVE AND NEGATIVE NUMBERS

18. The first numbers studied in arithmetic are the **integers,** such as, 1, 4, 15, etc.; the next are the common fractions, such as, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{7}{9}$, etc. and the decimals, such as, 2.03, 4.6, etc. The literal numbers in the first chapter represented only these same arithmetical numbers. One of the most distinctive things about algebra is its use of certain other numbers.

19. Opposite Quantities. Suppose that the temperature at a certain hour of the day was 73°, and that there was a change of 5°. To determine the new temperature, it would be necessary to know whether the change was a *rise* of 5° or the *opposite,* a *fall* of 5°.

Suppose that a person was known to weigh 85 pounds, and that during a certain time there was a change in his weight of 5 pounds. To determine his new weight, it is necessary to know whether the change was an *increase* of 5 pounds or the *opposite,* a *decrease* of 5 pounds.

These are two illustrations of opposite quantities. Many concrete quantities exist in two such opposite states.

EXERCISE 11

Tell the opposite of each of the following:

1. Sailing 35 miles north.
2. Sailing 25 miles east.
3. Receiving $30.
4. Gaining $5.
5. Moving 5 steps forward.
6. Depositing $15 in a bank.
7. Rise of 10° in temperature.
8. Walking 5 rods to the right.
9. Increasing weight by 5 pounds.
10. Adding 7.

11. What is the total result if any one of the changes in Examples 1 to 10 is followed by a second change of *opposite kind* and *of like amount ?*

12. What is the total result of two transactions, one giving a gain of $50, and one a loss of $25?

What single change will produce the same result as the two changes indicated in the following examples?

13. If a ship sails first 6° north and then 2° south?

14. If a ship sails first 8° east and then 10° west?

15. If a boy, becoming ill, loses 10 pounds, and then gains 8 pounds?

16. If the temperature first rises 12°, and then falls 15°?

17. If a man first deposits $100 in a bank and then withdraws $125 from his account?

18. A vessel sails from the equator due north 28°, and then due south 57°. What is her latitude at the end of the voyage?

20. In Example 10, Exercise 11, the opposite of adding 7 is subtracting 7. Addition is always indicated by the sign +, and subtraction by the sign −. These same signs are used to distinguish between opposite quantities. A quantity preceded by the + sign is called a **Positive Quantity**, and one preceded by the − sign is called a **Negative Quantity.** Quantities preceded by the signs plus and minus are called **Signed Quantities.**

EXERCISE 12

The following are naturally considered positive quantities. What are the corresponding negative quantities?

1. Sailing east.
2. Sailing north.
3. Right direction.
4. Increasing.
5. Rising temperature.
6. Forward.
7. Upward.
8. Deposits.

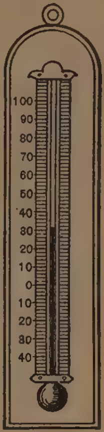
9. Assets. **10.** Profits. **11.** Above zero. **12.** Time A.D.

13. At 7 A.M. the temperature is -13°; at noon it is 8° warmer, and at 6 P.M. it is 5° colder than at noon. Required the temperature at noon and at 6 P.M.

14. At 7 A.M. the temperature is $+6^\circ$; at noon it is 14° colder, and at 6 P.M. it is 2° colder than at noon. Required the temperature at noon and at 6 P.M.

15. At 7 A.M. the temperature is -7°, and at noon $+9^\circ$. How many degrees warmer is it at noon than at 7 A.M.?

16. The temperature at 6 A.M. is -14°; during the morning it grows warmer at the rate of 3° per hour. Required the temperature at 9 A.M. and at 10 A.M.



17. The positive quantities in this set indicate rise in temperature. What single change will produce the same result?

(*a*) $+12^\circ$ and $+10^\circ$.

SOLUTION: 1. A 12° rise followed by a 10° rise gives a total of 22° rise.

2. This may also be expressed thus:

$$(+12^\circ)+(+10^\circ)=+22^\circ.$$

(*b*) $+9^\circ$ and -5°. (*c*) -4° and -5°. (*d*) $+7^\circ$ and -9°.

18. In this set the positive quantities refer to gains in financial transactions. What is the equivalent single change?

(*a*) $+\$15$, $+\$25$, $-\$30$.

SOLUTION: 1. A gain of \$15 followed by a gain of \$25 gives a gain of \$40; followed by a loss of \$30 gives a total result of \$10 gain.

2. Expressed in symbols thus:

$$(+\$15)+(+\$25)+(-\$30)=(+\$40)+(-\$30)=+\$10.$$

(*b*) $+\$20$, $+\$30$, $-\$50$. (*c*) $-\$45$, $+\$50$, $-\$10$.
(*d*) $+\$35$, $-\$20$, $-\$25$. (*e*) $+\$14$, $-\$20$, $+\$19$.

In the following problems, select and mark the positive and negative quantities. Find the total result as in the preceding problems and express it as a signed number.

19. A man's income during the year is $1500, and his expenses are $1300. Find the result at the end of the year.

20. A man's monthly account book shows the items: salary $150, rent $40, food $50, insurance $25, interest on savings $15. Find the result at the end of the month

21 Positive and Negative Numbers. The preceding exercises show that positive and negative quantities exist. In dealing with these quantities, positive and negative numbers are necessary.

Starting with an arithmetical number like 3, a new number called **Negative 3** is made; 3 is then called **Positive 3**. These two numbers are opposites and have the power of destroying each other when added, just as do opposite quantities which are equal in amount. Positive 3 is written, $+\mathbf{3}$; negative 3 is written, $-\mathbf{3}$ The arithmetical number 3 is called the **Absolute Value** of $+3$ and -3.

The negative number must always have its sign written before it; the positive number is often written without its sign.

Historical Note. — Hindu mathematicians, who knew about positive and negative numbers long before European mathematicians, in referring to them, used words which correspond to our words for assets and debits. They also were acquainted with the illustration of such numbers by means of the opposite directions on a straight line. To indicate that a number was a negative number, they placed a dot over it European mathematicians did not arrive at an equal understanding of positive and negative numbers until the sixteenth century.

EXERCISE 13

1. Read the following numbers, write each in symbols, and tell the absolute value of each:

(*a*) positive nine; (*b*) negative seven;
(*c*) positive four; (*d*) negative three fourths;
(*e*) negative five sixths;
(*f*) negative three and three tenths.

2. Read the following numbers and tell the absolute value of each:

(*a*) $+6$; (*b*) -2; (*c*) -5;
(*d*) $-\frac{2}{3}$; (*e*) $+\frac{3}{8}$; (*f*) -2.98;
(*g*) $+3.41$; (*h*) -45.087; (*i*) -102.34.

22. Addition of Positive and Negative Numbers. The rules for addition of signed numbers are suggested by the following problems:

1. Find the sum of $+5$ and $+3$.

Just as $5 gain plus $3 gain gives $8 gain, similarly

$$(+5)+(+3)=+8.$$

2. Find the sum of -5 and -3.

Just as $5 loss plus $3 loss gives $8 loss, similarly

$$(-5)+(-3)=-8$$

3. Find the sum of $+5$ and -3.

Just as $5 gain plus $3 loss gives $2 gain, similarly

$$(+5)+(-3)=+2.$$

4. Find the sum of -5 and $+3$.

Just as $5 loss plus $3 gain gives $2 loss, similarly

$$(-5)+(+3)=-2.$$

Rule. — 1. To add two positive numbers, add their absolute values (§ 21), and prefix the plus sign to the result.

2. To add two negative numbers, add their absolute values and prefix the minus sign to the result.

3. To add a positive and a negative number, find the difference of their absolute values, and prefix to the result the sign of the number having the greater absolute value.

EXERCISE 14

1. Find the sum of $+10$ and -3.

SOLUTION: Use Rule 3. Subtract 3 from 10; prefix + sign.

$$(+10)+(-3)=+7.$$

2. Find the sum of -12 and $+6$.

SOLUTION: Use Rule 3. Subtract 6 from 12; prefix − sign.

$$(-12)+(+6)=-6$$

3. Find the sum of -9 and -5.

SOLUTION: Use Rule 2. Add 9 and 5; prefix − sign.

$$(-9)+(-5)=-14.$$

4. To each of the numbers:

$$+6,\quad +9,\quad +14,\quad +5,\quad +18,\quad +3,$$

(*a*) add -4; (*b*) add -12; (*c*) add -15.

5. To each of the numbers:

$$-5,\quad -15,\quad -2,\quad -10,\quad -16,\quad -20,$$

(*a*) add $+6$; (*b*) add $+15$; (*c*) add $+12$;
(*d*) add -5; (*e*) add -6; (*f*) add -7.

6. To each of the numbers:

$$-36,\quad +48,\quad -17,\quad -25,\quad +24,\quad +29,$$

(*a*) add $+9$; (*b*) add -8; (*c*) add -7.

Find the sum:

✓ **7.**	**8.**	**9.**	**10.**	**11.**
$+124$	-15.2	-1.35	$+2.10$	-10.3
-36	$+9.1$	-1.63	-1.43	$+3.3$

12. $\begin{array}{r} +9 \\ \underline{-17} \\ -8 \end{array}$ Here 9 must be taken from 17. It is necessary to become expert in subtracting the upper number from the lower.

13.	**14.**	**15.**	**16.**	**17.**	**18.**
-11	$+17$	-28	$+72$	$+65$	-59
$+43$	-42	$+44$	-109	-247	$+78$

19. $+6$
-2
$+4$
-5

Hint: There are two ways of doing this example. Either add in order from bottom to top, or from top to bottom; or, add first all of the positive numbers and all of the negative numbers separately, and then combine the results.

20.	**21.**	**22.**	**23.**	**24.**	**25.**
$+4$	-7	$+15$	-5	$+16$	-10
-6	-5	-12	-14	-11	$+13$
-2	$+9$	$+6$	$+11$	-7	-14
$+5$	-1	$+6$	$+3$	$+22$	$+7$
-3	$+8$	-9	-8	-9	-3

Find the sum:

26. $-\frac{3}{4}, +\frac{1}{2}$.

27. $-\frac{3}{5}, -\frac{9}{10}$.

28. $-\frac{5}{3}, -\frac{1}{6}$.

29. $-\frac{9}{8}, +\frac{8}{7}$.

30. $+2\frac{1}{2}, -1\frac{3}{4}$.

31. $-3\frac{1}{3}, +2\frac{2}{3}$.

32. Demosthenes was born in the year -385, and died at the age of 63. What was the year of his death?

33. Pythagoras was born in the year -580, and lived to the age of 79 years. What was the year of his death?

34. At the beginning of the month the number of pupils in a schoolroom is 53; 3 pupils enter during the month and 5 leave school. How many pupils are there in the room at the end of the month?

35. A principal of a school finds that his six algebra classes have the enrollment indicated in line 2. He decides to make the changes indicated in line 3.

Classes	1	2	3	4	5	6
Members	29	15	22	28	18	14
Changes	-8	$+6$	-1	-7	$+3$	$+7$

(*a*) Tell what each of the changes in line 3 means.

(*b*) Find the result in the class membership.

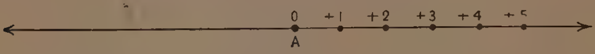
(*c*) How can he check his work from line 3?

23. From the examples in Exercise 14 it is clear that adding a negative number has the same effect as subtracting the positive number of equal absolute value.

Thus, $(+12)+(-5)=12-5=7.$

$(-10)+(-5)=-10-5=-15.$

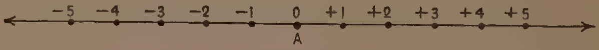
24. It is convenient to picture the positive numbers thus:



where $+1$ is placed at any distance from the point A, $+2$ twice as far, etc. In this sense, the positive numbers form a *scale* extending to the right. Any number precedes all larger numbers on this scale; thus, 3 precedes 4, 5, 6, etc.

Since $(-3)+(+3)=0$, it is natural to think of -3 as being 3 less than zero; and, similarly, of -4 as being 4 less than zero.

It is natural to think of the negative numbers as arranged on the left.



Thus, the positive and negative numbers together form a *complete scale* extending in both directions from zero.

Starting on the left, any negative number precedes the positive numbers and *may be thought of as being less than the positive numbers*

MULTIPLICATION OF POSITIVE AND NEGATIVE NUMBERS

25. The terms **Multiplier, Multiplicand,** and **Product** have the same meaning in algebra as in arithmetic.

The rules for multiplication of signed numbers are suggested by the following problems. In these problems, read the sign, ×, "multiplied by."

1. Find the product of $+4$ and $+3$.

Since positive numbers are like arithmetical numbers,

$$(+4) \times (+3) = +12.$$

2. Find the product of -4 and $+3$.

In arithmetic, to multiply by 3 means to add the multiplicand three times. If this is done in this problem,

$$(-4) \times (+3) = (-4) + (-4) + (-4) = -12.$$

3. Find the product of $+4$ and -3.

In arithmetic, $4 \times 3 = 3 \times 4$, since each equals 12. The order of the factors may be changed. If it is assumed that the same law holds in algebra,

$(+4) \times (-3)$ should equal $(-3) \times (+4)$ or, -12, by problem 2.

Then, $$(+4) \times (-3) = -12.$$

To multiply a number by a negative number seems to be accomplished by multiplying it by the absolute value of the multiplier and changing the sign of the product.

4. Find the product of -4 and -3.

The multiplier is a negative number. If the suggestion from problem 3 is followed, the result in this problem may be obtained by multiplying (-4) by 3 and changing the sign of the product.

$(-4) \times 3 = -12$. Changing the sign of -12 gives $+12$.

Therefore, $$(-4) \times (-3) = +12.$$

Gathering together the results of problems 1, 2, 3, and 4:

1. $(+4) \times (+3) = +12.$

2. $(-4) \times (+3) = -12.$

3. $(+4) \times (-3) = -12.$

4. $(-4) \times (-3) = +12.$

Rule. — **To multiply one signed number by another:**

1. Find the product of their absolute values. (See all four above.)

2. Make the product *positive* if the multiplicand and multiplier have *like* signs. (See 1 and 4 above.)

3. Make the product *negative* if the multiplicand and multiplier have *unlike* signs. (See 2 and 3 above.)

EXERCISE 15

1. Multiply the numbers:

$$+5,\quad +6,\quad +8,\quad +7,\quad +9,\quad +12,$$

(*a*) by $+4$; (*b*) by -3; (*c*) -8; (*d*) by -9.

2. Multiply the numbers:

$$-6,\quad -10,\quad -8,\quad -7,\quad -15,\quad -12,$$

(*a*) by $+4$; (*b*) by -9; (*c*) by -6; (*d*) by $+7$.

3. Multiply the numbers:

$$+11,\quad -9,\quad +14,\quad +12,\quad -25,\quad -15,$$

(*a*) by -6; (*b*) by $+7$; (*c*) by -8; (*d*) by -10.

Find the products of the following factors:

4. $-9, -11$.
5. $-7, +12$.
6. $+9, -20$.
7. $+6, -13$.
8. $-13, +8$.
9. $-9, +12$.
10. $-11, -20$.
11. $+9, -16$.
12. $+\frac{1}{2}, -\frac{1}{3}$.
13. $-\frac{1}{4}, +\frac{1}{5}$.
14. $-\frac{2}{5}, -\frac{1}{2}$.

15. Find the product of $-2, +3, -4$.

$$(-2)\cdot(+3) = -6;\ (-6)\cdot(-4) = +24.$$

16. $-1, +3, -2$.
17. $+3, -4, -5$.
18. $-6, -2, -5$.
19. $+5, -8, -3$.
20. $-6, -3, -4$.
21. $-7, +2, -5$.
22. $-9, +5, -4$.
23. $+10, -7, +6$.

24. If the number of negative factors is even, what is the sign of the product?

25. If the number of negative factors is odd, what is the sign of the product?

26. Powers of Positive and Negative Numbers.

$$(+2)^2 = (+2)(+2) = +4. \qquad (\S\ 15)$$
$$(+2)^3 = (+2)(+2)(+2) = +8.$$
$$(+2)^4 = (+2)(+2)(+2)(+2) = +16.$$

It is clear that every power of a positive number is positive.

$$(-2)^2 = (-2)(-2) = +4.$$
$$(-2)^3 = (-2)(-2)(-2) = -8.$$
$$(-2)^4 = (-2)(-2)(-2)(-2) = +16.$$
$$(-2)^5 = (-2)(-2)(-2)(-2)(-2) = -32.$$

It is clear that every *even* power of a negative number is *positive* and every *odd* power is *negative.*

EXERCISE 16

Find the values of the following powers:

1. $(-3)^2$.
2. $(-3)^5$.
3. $(+4)^3$.
4. $(-1)^5$.
5. $(-5)^3$.
6. $(-2)^6$.
7. $(-4)^4$.
8. $(+1)^6$.
9. $(-3)^4$.
10. $(-5)^2$.
11. $(-6)^3$.
12. $(-1)^6$.

13. Find the value of $ax^2 + bx + c$
when $a = 1,\ b = 2,\ c = -3,\ x = -2.$

$$\begin{aligned} ax^2 + bx + c &= 1 \cdot (-2)^2 + (+2)(-2) + (-3) \\ &= +4 - 4 - 3 = -3. \end{aligned}$$

In the following examples, let $a = +1$, $b = -2$, $c = +3$, and $x = -2$. Find the values of the expressions:

14. $3\,ab$.
15. ab^2.
16. $5\,bx^2$.
17. $-11\,bcx$.
18. $bc + ax$.
19. $a^2b + ab^2$.
20. $bx^2 - cx^2$.
21. $ax^3 - bx^3$.
22. $ab^3 + bc^3$.
23. $b^3 + x^3$.
24. $a^4 - x^4$.
25. $a^4b^3 - a^3c$.

III. ADDITION AND SUBTRACTION OF ALGEBRAIC EXPRESSIONS

DEFINITIONS

27. A **Monomial** or **Term** is an expression whose parts are not separated by the signs + or −. Thus,

$2x^2$, $-3ab$, and $+5$ are the *terms* of the expression $2x^2 - 3ab + 5$.

In an expression, a term whose sign is plus may be called a *positive term*, and one whose sign is minus, a *negative term*, as the terms $2x^2$, and $-3ab$, respectively, although the algebraic value of the term depends upon the values of its literal factors.

28. If two or more numbers are multiplied together, each of them, or the product of any number of them, is called a **Factor** of the product.

Thus, a, b, c, ab, ac, and bc are factors of the product abc.

29. Any factor of the product is called the **Coefficient** of the product of the remaining factors.

Thus, in $2ab$, 2 is the coefficient of ab, $2a$ of b, a of $2b$, etc.

30. If one factor of a product is expressed in *numerals* and the other factor is expressed in *letters*, the former is called the **Numerical Coefficient** of the latter.

Thus, in $2ab$, 2 is the numerical coefficient of ab.

If no numerical coefficient is expressed, the coefficient 1 is understood; thus, a is the same as $1a$.

31. By § 25, $(-3) \times a = -3a$; that is, $-3a$ is the product of -3 and a. Then -3 is the numerical coefficient of a in $-3a$.

Thus, in a negative term, the numerical coefficient includes the sign.

32. Terms which are alike in their literal parts are called **Like or Similar Terms**; as, $2x^2y$ and $-7x^2y$.

Terms which are not alike in their literal parts are called **Unlike or Dissimilar Terms**; as, $2x^2$ and $7xy^2$.

Sometimes unlike terms may be *like with respect to one or more letters.* Thus, $2axy$ and $3bxy$ are like with respect to xy.

EXERCISE 17

1. Tell all of the factors of the monomial $+6xy$

2. What is the numerical coefficient of the monomial $-6xy$?

3. In 2, what is the coefficient of y? of x? of xy? of $6x$?

4. Select the sets of like terms:

(*a*) $+2xy^2, -3xy^2$ (*b*) $+7m, -6n$,

(*c*) $+5abc, -6ab^2c$ (*d*) $-8ab, +9ab$.

33. The result obtained by adding two or more numbers is called the **Sum.**

ADDITION OF MONOMIALS

34. Addition of Like Terms. In paragraph 11, it was found that $6n+4n=10n$. $6n$ and $4n$ are *like* terms since they have the common factor n (§ 11). The coefficients of n are 6 and 4, and their sum is the coefficient, 10, of the result.

Rule. — To add two or more like terms:

1. Multiply their common factor by the sum of its coefficients.

EXAMPLE 1. Find the sum of $5x^2$ and $3x^2$.

SOLUTION: 1. The common factor in $5x^2$ and $3x^2$ is x^2. The coefficients are 5 and 3; their sum is 8.

2. Hence, $5x^2+3x^2=8x^2$.

CHECK. Let $x=-2$. $5x^2=5\cdot(-2)^2=5\cdot4=20$, $3x^2=3\cdot4=12$, and $20+12=32$. Also, $8x^2=8\cdot4=32$.

EXAMPLE 2. Find the sum of $-5x^2y$ and $+3x^2y$.

SOLUTION: $-5x^2y+3x^2y=[(-5)+(+3)]x^2y=-2x^2y$.

Check: Let $x = 1$ and $y = 1$. Then, $-5x^2y = -5 \cdot 1 \cdot 1 = -5$, $+3x^2y = +3 \cdot 1 \cdot 1 = +3$, and $(-5) + (+3) = -2$. Also, $-2x^2y = (-2) \cdot 1 \cdot 1 = -2$.

Example 3. Find the sum of 16×19 and 14×19.

Solution: 1. The common factor is 19; its coefficients are 16 and 14.
2. $16 \times 19 + 14 \times 19 = (16 + 14) \times 19 = 30 \times 19 = 570$.

EXERCISE 18

✓ Find the sum in each of the following:

1. $5A$ and $-12A$.
2. $11p$ and $-6p$.
3. $-7m$ and $-8m$.
4. $-4n$ and $-9n$.
5. $+15E$ and $-11E$.
6. $-5ab$ and $+13ab$.
7. $-13r^2s$ and $+36r^2s$.
8. $-17x^2$ and $-15x^2$.
9. xyz and $-9xyz$.
10. $8x^3y^2$ and $-29x^3y^2$.
11. $-bc$ and $+6bc$.
12. 12×16 and 8×16. 320
13. 21×17 and 9×17. 510
14. 13×23 and 7×23. = 460

15.	16.	17.	18.	19.
$+7a$	$+15xy$	$+12r^2s$	$-2xy^2$	$+5m$
$+9a$	$-4xy$	$-3r^2s$	$-9xy^2$	$-15m$

20.

$$\begin{array}{r} -7a \\ +9a \\ +8a \\ -3a \\ \hline +7a \end{array}$$

Hint. Add first the positive terms, getting $+17a$; then the negative, getting $-10a$; then add these results, getting $+7a$. All should be done mentally.

21.	22.	23.	24.	25.
$15m$	$-13x^2$	$+8ab$	$16xyz$	$21ab^2c$
$-2m$	$+2x^2$	$-7ab$	$-4xyz$	$-6ab^2c$
$-5m$	$-7x^2$	$-5ab$	$-6xyz$	$-17ab^2c$
$+3m$	$+3x^2$	$+3ab$	$+3xyz$	$+ab^2c$
			+9	

26. Find the sum of $9a^3$, $-7a^3$, and $+8a^3$.

SOLUTION: $9a^3 + (-7a^3) + 8a^3 = 9a^3 - 7a^3 + 8a^3 = 10a^3$.

NOTE. This illustrates another arrangement of an addition example. The terms are first connected by the + sign. Then $+(-7a^3)$ is written as $-7a^3$, since adding a negative number is the same as subtracting the positive number of equal absolute value. (§ 21.)

Add, as in Example 26, the following:

27. $7ab$, $-3ab$, and $+9ab$. **29.** $12y$, $-7y$, $+9y$, and $-8y$.

28. $9r^3$, $-6r^3$, $+3r^3$, and $10r^3$. **30.** $15z$, $+9z$, $-11z$, and $-8z$.

31. Find the sum of $12a$, $-5x$, $-3y^2$, $-5a$, and $8x$.

SOLUTION:

$$\begin{aligned} &12a + (-5x) + (-3y^2) + (-5a) + (8x) \\ &= 12a - 5x - 3y^2 - 5a + 8x. \\ &= 7a + 3x - 3y^2. \end{aligned}$$

In this Example, $12a$ and $-5a$ give $7a$; $-5x$ and $+8x$ give $+3x$; there is no term to combine with $-3y^2$. Only the like terms may be combined.

Add, as in Example 31, the following:

32. $8ab$, $-9cd$, $-6ab$, and $+4cd$.

33. $6x^2$, $-10z^2$, $2y^2$, $4z^2$, $-9y^2$, and $-3x^2$.

34. $12r^2$, $-3r$, $-8s$, $-5r$, $+3s$, $-7r^2$, and $11s$.

35. $10c$, $-4d$, $-3k$, $+9c$, $-4k$, $+5d$, $-6k$, and $-2c$.

36. $11m^3$, $-8n^2$, $+6l$, $-5n^2$, $+3m^3$, and $-8l$.

Simplify the following:

37. $5a - 8a + 10a - 12a$.

38. $-3x^2 + 6x^2 + x^2 - 11x^2 + x^2$.

39. $4r^2s - 7r^2s + 5r^2s - 6r^2s$.

40. $-8m^2n - 3m^2n + 2m^2n - 9m^2n$.

41. $6c + 11d - 4d - 5c - 2c$.

42. $2x^2 - 3x - 4x - 5x^2 + 8x$.

43. $9r^3 + 2r^2 - 3r^3 - 4r^2.$

44. $\frac{1}{2}a + \frac{1}{3}a - \frac{1}{6}a.$

45. $-\frac{1}{4}mn + \frac{1}{8}mn + \frac{1}{2}mn.$

46. $-\frac{2}{3}r^2s^2 + \frac{1}{6}r^2s^2 - \frac{1}{2}r^2s^2.$

47. $\frac{3}{7}xy - \frac{1}{14}xy - \frac{1}{2}xy.$

48. $\frac{1}{5}y^3 + \frac{3}{10}y^3 - y^3 + \frac{1}{2}y^3.$

49. $2.5xy - .3xy + 1.2xy.$

50. $-3.05a + 4.4a - 1.3a.$

DEFINITIONS

35. A **Polynomial** is an algebraic expression consisting of two or more terms: as, $a + b$, or $2x^2 - 3xy + 5y^2$

A **Binomial** is a polynomial of two terms: as, $a + b$.

A **Trinomial** is a polynomial of three terms.

ADDITION OF POLYNOMIALS

36. Addition of polynomials is similar to addition of denominate numbers in arithmetic.

EXAMPLE 1. Find the sum of 3 yd. 2 ft. and 6 in. and 5 yd. 1 ft. and 4 in.

SOLUTION:

$$\begin{array}{r} 3\text{ yd.} + 2\text{ ft.} + 6\text{ in.} \\ \underline{5\text{ yd.} + 1\text{ ft.} + 4\text{ in.}} \\ 8\text{ yd.} + 3\text{ ft.} + 10\text{ in.} \end{array}$$

EXAMPLE 2. Find the sum of $2a+3b+5c$ and $4a+b+2c$.

SOLUTION:

$$\begin{array}{r} 2a + 3b + 5c \\ \underline{4a + b + 2c} \\ 6a + 4b + 7c \end{array}$$

Rule. — To add two polynomials:

1. Rewrite the polynomials, if necessary, so that like terms are in the same vertical column.

2. Add the columns of like terms.

3. Write the results of step 2 with their proper signs for the sum.

EXAMPLE 1. Find the sum of $6a - 7x^2$, $3x^2 - 2a + 3y^2$, and $2x^2 - a - mn$.

SOLUTION:

$$\begin{array}{rrrr} 6a & -7x^2 & & \\ -2a & +3x^2 & +3y^2 & \\ \underline{-a} & \underline{+2x^2} & \underline{} & \underline{-mn} \\ 3a & -2x^2 & +3y^2 & -mn \end{array}$$

Check: Letting $a = 2,\ x = 1,\ y = 1,\ m = 2,\ n = 1$:

$$6a - 7x^2 = 12 - 7 = 5;\ -2a + 3x^2 + 3y^2 = -4 + 3 + 3 = +2;$$

$$-a + 2x^2 - mn = -2 + 2 - 2 = -2;$$

and the sum of these values of the polynomials is $5 + 2 - 2$ or 5.

The value of the sum of the polynomials,

$$3a - 2x^2 + 3y^2 - mn \text{ is } 6 - 2 + 3 - 2 \text{ or } 5.$$

Since the two values are equal, the solution is probably correct. This is called **checking by substitution.**

Another method of checking is to add carefully in the opposite direction, as in arithmetic.

Example 2. Find the sum of $x^3 - y^3 + 3xy^2 - 3x^2y$, $2x^3 + 3xy^2 + y^3$, and $2x^2y - 4xy^2$. Check for $x = 1$, $y = 1$.

Solution: 1.

$$\begin{array}{rrrrcrrrrcr} x^3 & -3x^2y & +3xy^2 & -y^3 & = & 1 & -3 & +3 & -1 & = & 0 \\ 2x^3 & & +3xy^2 & +y^3 & = & 2 & & +3 & +1 & = & 6 \\ & +2x^2y & -4xy^2 & & = & & 2 & -4 & & = & -2 \\ \hline 3x^3 & -x^2y & +2xy^2 & & & & & & & & +4 \end{array}$$

Also, $3x^3 - x^2y + 2xy^2 = 3 - 1 + 2 = +4$. The solution is correct.

37. In Example 2 of this last paragraph, notice that the three polynomials have been rearranged. The term containing the highest power of x is placed first; then the term having the next lower power of x, and so on. The term $-y^3$, not containing any x, is thought of as containing the lowest power of x in this expression. The polynomials are arranged in **Descending Powers** of x.

These polynomials are arranged also in **Ascending Powers** of y. The exponents of y in the terms increase from left to right.

In all examples in addition, subtraction, multiplication, and division of polynomials, it is advisable to arrange the polynomials, if necessary, according to ascending or descending powers of one of the letters when writing them preparatory to solving the example. When the polynomials are so arranged, there is less likelihood of making some error in the solution, comparison and checking of results is facilitated, and the final solution has a more workmanlike and finished appearance.

EXERCISE 19

Find the sums in the following:

1.

$$\begin{array}{rr} 7\,A & -\,3\,B \\ -9\,A & +\,2\,B \\ +3\,A & -\,B \\ \hline \end{array}$$

2.

$$\begin{array}{rr} -\,6\,t^2 & +\,5\,n^3 \\ +12\,t^2 & -\,13\,n^3 \\ -\,8\,t^2 & +\,14\,n^3 \\ \hline \end{array}$$

3.

$$\begin{array}{rr} -16\,am & +\,14\,bn \\ 6\,am & -\,5\,bn \\ 9\,am & -\,5\,bn \\ \hline \end{array}$$

4.

$$\begin{array}{rrr} a^2 & -\,2\,ab & +\,b^2 \\ a^2 & +\,4\,ab & -\,2\,b^2 \\ -\,2\,a^2 & -\,2\,ab & \\ \hline \end{array}$$

5.

$$\begin{array}{rrrr} 7\,r^3 & & -\,6\,rs^2 & \\ & +\,6\,r^2s & & -\,10 \\ -\,11\,r^3 & -\,5\,r^2s & +\,15\,rs^2 & \\ \hline \end{array}$$

6.

$$\begin{array}{rrrr} 3\,x^3 & -\,5\,x^2y & +\,2\,xy^2 & -\,4\,y^3 \\ & +\,6\,x^2y & & +\,5\,y^3 \\ -\,5\,x^3 & & -\,7\,xy^2 & \\ \hline \end{array}$$

7.

$$\begin{array}{rrrr} -\,7\,m^3 & & +\,5\,mn^2 & \\ & -\,6\,m^2n & -\,13\,mn^2 & +\,2\,n^3 \\ & +\,4\,m^2n & +\,3\,mn^2 & \\ \hline \end{array}$$

NOTE. In Examples 5 and 7, blank spaces are left when the first polynomial is written for powers which are not present in the polynomial.

8. $3\,a - 4\,b + 6\,c$ and $5\,a + 7\,b - 3\,c$.

9. $4\,k - 7 + 3\,m$, $5\,k + 2 - 4\,m$, and $3\,k - 2$.

10. $2\,a + 8\,b - 5\,c$, $-3\,b - 3\,c + 7\,d$, and $+6\,c - 4\,a + 2\,d$.

11. $12\,r + 6\,s - 9\,t$, $8\,r - 9\,s + 11\,t$, and $15\,r + 7\,s - 6\,t$.

12. $x^2 + 2\,xy + y^2$, $x^2 - 2\,xy + y^2$, and $-2\,x^2 - 2\,y^2 + 10\,xy$.

13. $4\,m^2 - 4\,mn + n^2$, $m^2 + 4\,mn + 4\,n^2$, and $-5\,m^2 + 5\,n^2$.

14. $a^3 + 3\,a^2b + 3\,ab^2 + b^3$ and $a^3 - 3\,a^2b + 3\,ab^2 - b^3$.

15. $3\,x^2 + 9\,xy - 4\,y^2$, $-2\,xy - 5\,x^2 - 10\,y^2$, and $7\,y^2 - 6\,xy + 8x^2$.

16. $15\,a^3 - 6 + 4\,a + 16\,a^2$ and $-2 - 13\,a + 8\,a^2 - 3\,a^3$.

17. $x^5 + x^2y^3 + x^3y^2 + x^4y + xy^4 + y^5$ and $x^5 - x^4y + xy^4 + x^3y^2 - y^5$.

18. $12\,x^3 - 4\,x + 3$, $-8\,x + 6 + 15\,x^3 + 2\,x^2$, and $-16 + 6\,x^2 - 8\,x^3 - 11\,x$.

[illegible]

work : prove

19. $9A^3 + 3AB^2 - B^3$, $-2A^2B + 5AB^2 + 7B^3$, and $-6A^3 + 13A^2B$.

20. $\frac{1}{2}a + \frac{1}{3}b - \frac{1}{5}c$ and $\frac{1}{4}a - \frac{1}{6}b + \frac{1}{10}c$.

21. $\frac{2}{3}m - \frac{3}{5}n + r$, $-\frac{1}{3}m + \frac{3}{10}n - \frac{1}{2}r$, and $2m + \frac{1}{10}n - \frac{1}{2}r$.

22. $3a - \frac{1}{2}b - \frac{9}{14}c$ and $-\frac{3}{4}a - \frac{1}{6}b + \frac{5}{7}c$.

23. $2.3A - 6.02B + 3.5C$ and $-1.6A - 4.38B - 2C$.

24. $2.25x^2 - 3.5xy + 4y^2$ and $-1.5x^2 + 2.75xy - 3.2y^2$.

25. $3m - 1.3n + 4$ and $.5m + 2n - 1.6$.

26. $2a^2 - 5ab - b^2$, $7a^2 + 3ab - 9b^2$, and $-4a^2 - 6ab + 8b^2$.

27. $4x - 3x^2 - 11 + 5x^3$, $12x^2 - 7 - 8x^3 - 15x$, and $14 + 6x^3 + 10x - 9x^2$.

28. $x^3 - 3xy^2 - 2x^2y$, $3x^2y - 5y^3 - 4xy^2$, $5xy^2 - 6y^3 - 7x^3$, and $8y^3 + 7x^3 - 9x^2y$.

29. $15a^3 - 2 - 9a^2 - 3a$, $13a - 5a^2 - 6 - 7a^3$, $8 + 4a - 8a^3 - 7a^2$, and $16a^2 + 3a^3 - 10a - 2$.

30. $12a^3 - x^3 + 4ax^2 - 5a^2x$, $18x^3 - 2a^2x - 3a^3 - 13ax^2$, $15a^2x + 11x^3 - 17a^3 + 3ax^2$, and $6ax^2 - 8a^2x - 7x^3 + 9a^3$.

SUBTRACTION

38. Subtraction is the process of finding what number must be added to one given number to produce another given number.

Thus, subtracting 3 from 8 determines the number which must be added to 3 to give 8; and subtracting a from b determines the number which must be added to a to give b.

The number subtracted is called the **Subtrahend.**

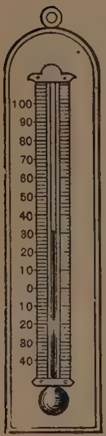
The number from which the subtrahend is subtracted is called the **Minuend.**

The result is called the **Remainder** or **Difference.**

EXERCISE 20

What must be added to the first number to give the second?

1. $4, 12.$
2. $11, 15.$
3. $12, 18.$
4. $3a, 5a.$
5. $4m, 7m.$
6. $8m, 15m.$
7. $13c, 21c.$
8. $25b, 30b.$
9. $16x, 22x.$



How much must the temperature rise to change:

10. $+2^\circ$ into $+10^\circ$?
11. $+3^\circ$ into $+12^\circ$?
12. -8° into -3°?
13. -6° into -2°?
14. -3° into $+2^\circ$?
15. -6° into $+4^\circ$?
16. -5° into $+7^\circ$?
17. -2° into $+9^\circ$?

Find the number to be added:

18. $(+3)+?=+8.$
19. $(-3)+?=0.$
20. $(-3)+?=+2.$
21. $(-3)+?=+4.$
22. $(-4)+?=+6.$
23. $(-5)+?=+3.$
24. $(-8)+?=-2.$
25. $(-6)+?=+6.$

How much must the temperature fall to change:

26. $+2^\circ$ into 0°?
27. $+2^\circ$ into -3°?
28. $+4^\circ$ into -6°?
29. $+3^\circ$ into -5°?
30. $+8^\circ$ into -3°?
31. -2° into -8°?
32. $+10^\circ$ into -3°?
33. 0° into -8°?

Find the number to be added:

34. $(+3)+?=0.$
35. $(+3)+?=-2.$
36. $(+4)+?=-7.$
37. $(+5)+?=-6.$
38. $(+2)+?=-5.$
39. $(+1)+?=-7.$
40. $(+7)+?=-2.$
41. $(-5)+?=-7.$
42. $(-4)+?=-8.$
43. $(-1)+?=-9.$
44. $(-2)+?=-10.$
45. $(-3)+?=-12.$

39. Subtraction of Positive and Negative Numbers.

The rule for subtraction of signed numbers is suggested by the following problems:	(A) SIGNS AS GIVEN; SUBTRACT	(B) SIGNS CHANGED; ADD
1. Subtract +2 from +6 This means: (+2) + ? = +6. Result, +4 Hence, (+6) − (+2) = +4.	+6 +2 +4	+6 −2 +4
2. Subtract −2 from +6. This means: (−2) + ? = +6. Result, +8. Hence, (+6) − (−2) = +8.	+6 −2 +8	+6 +2 +8
3. Subtract +2 from −6. This means: (+2) + ? = −6. Result, −8. Hence, (−6) − (+2) = −8.	−6 +2 −8	−6 −2 −8
4. Subtract −2 from −6. This means: (−2) + ? = −6. Result, −4. Hence, (−6) − (−2) = −4.	−6 −2 −4	−6 +2 −4

In the column *A* on the right, the problem is arranged as usual for a subtraction problem, with the result as it was obtained in the solution on the left; in the column *B* on the right, *the sign of the subtrahend has been changed* from + to −, or from − to +. Notice that the correct result is then obtained by *adding*. Hence,

Rule. — **To subtract one number from another, change the sign of the subtrahend and add it to the minuend.**

HISTORICAL NOTE. — The symbol, −, like the symbol, +, first appeared in print in a mathematical book by Widmann in 1489. The Italian and French mathematicians of the same period used the symbol, m̃, derived from the first letter of the Latin word *minus*

EXERCISE 21

1. Subtract -6 from -15.

SOLUTION: $\begin{array}{r} -15 \\ \underline{-\ \ 6} \\ -\ \ 9 \end{array}$ Imagine the sign of the subtrahend changed to + and then add by the rule in § 22. Do not change the sign on the paper.

2. Find the remainders:

$$\begin{array}{rrrrrrrr} +13 & +9 & +8 & -11 & -14 & -16 & -8 & -12 \\ \underline{+\ \ 4} & \underline{-5} & \underline{+3} & \underline{+\ \ 2} & \underline{-\ \ 6} & \underline{+\ \ 3} & \underline{+10} & \underline{+\ \ 9} \end{array}$$

3. $$\begin{array}{rrrrrrrr} -29 & -34 & +26 & -45 & +37 & -19 & -35 & +57 \\ \underline{+\ \ 7} & \underline{-\ \ 6} & \underline{-\ \ 5} & \underline{+\ \ 9} & \underline{+\ \ 8} & \underline{+\ \ 6} & \underline{+15} & \underline{+13} \end{array}$$

just (+ 3)

4. From each of the minuends of Example 3, subtract -11.

5. From each of the minuends of Example 3, subtract $+12$.

6. From each of the minuends of Example 3, subtract -14.

7. The minimum temperature on a certain day in Chicago was -14 and the maximum was -3. What was the range of temperature?

8. The Roman nation lived under a republican form of government from the year -509 to the year -31; for how many years did the republic last?

9. Was subtraction always possible in arithmetic?

10. Is subtraction always possible in algebra? Why?

11. Find the remainders:

$$\begin{array}{rrrrrr} +15\,a & -12\,x & -\ \ 7\,m & +14\,n^2 & +27\,rt & -32\,xyz \\ \underline{-\ \ 6\,a} & \underline{+18\,x} & \underline{+15\,m} & \underline{-\ \ 8\,n^2} & \underline{+34\,rt} & \underline{+18\,xyz} \end{array}$$

12. $$\begin{array}{rrrrr} -16\,r^2s & +33\,xy^2 & -26\,m^3 & -43\,n^2 & -45\,xy \\ \underline{-19\,r^2s} & \underline{-15\,xy^2} & \underline{-47\,m^3} & \underline{-17\,n^2} & \underline{+13\,xy} \end{array}$$

Subtract:

13. $-xy$ from $+xy$.

14. $-15\,a^3$ from $-46\,a^3$.

15. $21\,abc$ from $-39\,abc$.

16. $-45\,ax^4$ from $+19\,ax^4$.

17. Subtract $-31a^2b$ from $-8a^2b$.

18. Subtract $19rs$ from $-6rs$.

19. Take $8a$ from $-12a$.

20. From $-3m^2$ take $4m^2$.

21. Take $19a$ from $-23a$.

22. Take $-16n^3x$ from $-27n^3x$.

SUBTRACTION OF POLYNOMIALS

40. Rule. — To subtract one polynomial from another:

1. Rewrite the minuend, if necessary, in descending powers of some one letter.

2. Write below it the subtrahend, having like terms in the same vertical column.

3. Imagine the signs of the terms of the subtrahend changed, and add the resulting terms to those of the minuend.

EXAMPLE. Subtract $7ab^2 - 9a^2b + 8b^3$ from $-2a^2b + 4ab^2 + 5a^3$.

SOLUTION: 1. Arrange according to descending powers of a.

$$\begin{array}{lrrrrl}
\text{Minuend:} & 5a^3 & -2a^2b & +4ab^2 & & \\
\text{Subtrahend:} & & -9a^2b & +7ab^2 & +8b^3 & \text{Change signs mentally; add.} \\
\hline
\text{Remainder:} & 5a^3 & +7a^2b & -3ab^2 & -8b^3 &
\end{array}$$

CHECK: The sum of the subtrahend and remainder is the minuend.

Tues

EXERCISE 22

check 1st 5

Subtract; check each in some way:

1.

$$\begin{array}{r}
12a^2 - 9a - 7 \\
8a^2 - 6a + 13 \\
\hline
\end{array}$$

2.

$$\begin{array}{r}
5a - 8x + 2y \\
-3a + 7x - 3y \\
\hline
\end{array}$$

3.

$$\begin{array}{r}
17m - 12n + 6p \\
20m - 16n - 5p \\
\hline
\end{array}$$

4.

$$\begin{array}{r}
3a + 7b - c \\
2a + 4b - c \\
\hline
\end{array}$$

5.

$$\begin{array}{r}
2ab + 5bc - 3ac \\
-ab + 11bc - 4ac \\
\hline
\end{array}$$

6.

$$\begin{array}{r}
x^2 + 13x - 11 \\
-3x^2 + 6x - 5 \\
\hline
\end{array}$$

7.

$$\begin{array}{rrr} 10r^3 & & +6rs \\ 3r^3 & -2r^2s & +8rs \\ \hline \end{array}$$

8.

$$\begin{array}{rrrr} & 5x^2 & -6x & \\ x^3 & +2x^2 & & -5 \\ \hline \end{array}$$

9.

$$\begin{array}{rrrr} a^3 & +3a^2b & & +2b^3 \\ a^3 & -3a^2b & +ab^2 & -b^3 \\ \hline \end{array}$$

10.

$$\begin{array}{rrrr} x^5 & & & +y^5 \\ x^5 & -x^4y & +xy^4 & +y^5 \\ \hline \end{array}$$

11.

$$\begin{array}{rrr} x^2 & -2xy & +y^2 \\ x^2 & +2xy & +y^2 \\ \hline \end{array}$$

12.

$$\begin{array}{rrrr} x^3 & +x^2y & +xy^2 & +y^3 \\ x^3 & -x^2y & +xy^2 & -y^3 \\ \hline \end{array}$$

13. From $5a - 3b + 4c$ subtract $5a + 3b - 4c$.

14. From $-2m^2 - 4mn + 9n^2$ subtract $8m^2 - 7mn + 14n^2$.

15. From $ab + bc + ac$ subtract $ab - bc + ac$.

16. From $4x^3 - 9x^2 - 11x + 18$ subtract $3x^3 - 8x^2 - 17x + 25$.

17. From $-3y + 8x - 4z$ subtract $-z + 11x - 6y$.

18. From $8A + 2B - 7C$ subtract $8A - 2B + 7C$.

19. Subtract $2x^3 - 7 - 4x - 6x^2$ from $5x^2 - 12 + 9x^3 - 2x$.

20. From $x^2 - 2xy + y^2$ subtract $x^2 - 2xy - y^2$.

21. Take $7b - 9c - 2d$ from $6a - 5b + 12c$.

22. Take $12a^3 + 4a - 9$ from $3a^3 + 8a^2 - 6$.

23. Subtract $1 + a^3 - a - a^2$ from $3a - 3a^2 + 1 - a^3$.

24. From $10x^3 - 21x^2 - 11x$ take $-15x^2 - 20x + 12$.

25. From $17a^3 - 12ab^2 + 5b^3$ take $8a^3 - 3a^2b + 13b^3$.

26. Take $6c - 5d - 9b - 4a$ from $-10b - 2c + 3a - 9d$.

27. Subtract $4 - 3x - x^2 + 8x^3 + 10x^4$ from $9 - 7x + 6x^2 - 12x^3 + 5x^4$.

28. From $7a - 11a^3 - 8 + 6a^5$ subtract $16a^2 - 9 + 2a^5 + 15a - 10a^4$.

29. From $x^5 + 3x^4y - x^3y^2 + 5x^2y^3 - 4xy^4$ subtract $8x^4y - 7x^3y^2 - 6x^2y^3 + 11xy^4 - y^5$.

30. Subtract

$7n^2 - 5 - 20n^3 + 13n$ from $-9 - 14n^2 + 16n + 5n^3$.

31. Subtract
$-x^3+3x^2y-3xy^2+y^3$ from $x^3-2x^2y-2xy^2+y^3$.

32. Subtract $-2x^2-13+41x^3$ from $x+15x^3-18$.

33. Subtract $\frac{1}{2}m-\frac{1}{3}n+\frac{2}{7}p$ from $\frac{3}{4}m+\frac{2}{3}n+\frac{3}{14}p$.

34. Subtract $\frac{5}{6}a-\frac{1}{12}b+\frac{3}{10}c$ from $2a-\frac{1}{2}b-\frac{2}{5}c$.

35. Subtract $-\frac{3}{5}v+\frac{1}{6}w-\frac{3}{7}x$ from $\frac{2}{15}v+\frac{2}{3}w-\frac{3}{14}x$.

36. By how much does $81b^2+4a^2-36ab$ exceed $-30ab+9a^2+25b^2$?

37. By how much does $-5c+12a-8b$ exceed $7a-9c-b$?

38. By how much does 0 exceed $-3a+2b-c$?

39. By how much does 1 exceed $-5a-4b+6$?

40. From $a^2-2ab+b^2$ subtract the sum of $-a^2+2ab-b^2$ and $-2a^2+2b^2$.

HINT: The last two expressions are both to be subtracted; they are therefore subtrahends and should have their signs changed. Write down the minuend, and, below it, the two subtrahends with their signs changed; then add, thus doing the whole example at one operation.

41. From the sum of $3a^2-2ab+b^2$ and $5a^2-8ab+6b^2$ take $6a^2-4ab-3b^2$.

42. From $9x^2-8x+x^3$ take the sum of $5-x^2+x$ and $6x^5-7x-4$.

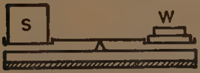
43. From the sum of $x+y-7z$ and $+4x-9y$ take the sum of $9x-2y+z$ and $-5x+6y-7z$.

44. From the sum of $2a+3b-4d$ and $2b-4c+3d$ take the sum of $4a-4b-3c+2d$ and $3a+2c$.

45. Subtract $.5x+.25y-1.2z$ from $3x-1.75y+.8z$.

41. Addition and Subtraction used in Equations. There are two more important rules used in solving equations. They will be illustrated by the scales. Review §§ 9 and 10.

Suppose that the sugar S exactly balances the weight W on the scales in the figure. If a 3-lb. weight be added to the right scalepan, and 3 lb. of sugar to the left scalepan, then the scales will still balance.



Similarly, if any number of pounds of weight be removed from the right scalepan and an equal weight of sugar from the left scalepan, then the scales will still balance.

These facts illustrate the rules:

Rule. — 1. The same number may be added to both members of an equation without destroying the equality.

2. The same number may be subtracted from both members of an equation without destroying the equality.

Example 1. Solve the equation $x - 3 = 7$.

Solution: 1. $x - 3$ is 3 less than x; if 3 is added to $x - 3$, the sum is therefore x. Add 3 to both members of the equation in order to keep them equal.

2. Adding 3 to both members, $x - 3 + 3 = 7 + 3$,
3. or $x = 10$.

Check: Does $10 - 3 = 7$? Yes.

Example 2. Solve the equation $18x - 5 = 3x + 55$.

Solution: 1. $18x - 5 = 3x + 55$.

2. Adding 5 to both members of the equation,

$$18x = 3x + 60.$$

3. Subtracting $3x$ from both members of the equation,

$$15x = 60.$$

4. Dividing both members of the equation by 15,

$$x = 4.$$

Check: Substitute 4 in equation 1; does $18 \times 4 - 5 = 3 \times 4 + 55$? does $72 - 5 = 12 + 55$? Yes.

42. In order to abbreviate the explanation of the solutions of equations, symbols A, S, M, and D will be used.

Thus: A_3 will mean "add 3 to both members of the equation."

S_{2n} will mean "subtract $2n$ from both members of the equation."

M_{-3} will mean "multiply both members of the equation by -3."

D_7 will mean "divide both members of the equation by 7."

These symbols will be used in the text from now on.

Pupils will find them helpful when solving equations.

EXERCISE 23

Tell what the following symbols mean:

1. A_2.
2. S_{-8}.
3. M_{5x}.
4. A_{-8y}.
5. D_{-4}.
6. M_{-1}.
7. A_{2c}.
8. D_{34}.
9. S_{7r}.
10. M_{-5}.

11. Solve the equation $24 - 11\,m = 6 - 8\,m$.

SOLUTION: 1.	$24 - 11\,m = 6 - 8\,m.$	
2. A_{8m}:	$24 - 3\,m = 6.$	(Rule 1, § 41)
3. S_{24}:	$-3\,m = -18.$	(Rule 2, § 41)
4. M_{-1}:	$+3\,m = +18.$	(Rule, § 10)
5. D_3:	$m = 6.$	(Rule, § 9)

CHECK: Substitute 6 in the given equation.

Does $24 - 11 \times 6 = 6 - 8 \times 6$? does $24 - 66 = 6 - 48$? does $-42 = -42$? Yes.

NOTE 1. In step 2, "A_{8m}" means "add $8\,m$ to both members of the previous equation"; the result will be equation 2. In step 4, "M_{-1}" means "multiply both members of the previous equation by -1"; the result will be equation 4.

NOTE 2. Whenever the coefficient of the unknown is negative, as in step 3, multiply both members by -1 so as to make it positive.

Solve the following equations and test the result:

12. $x + 3 = 12.$
13. $y + 7 = 15.$
14. $m - 2 = 9.$ ✓
15. $2\,m + 5 = 11.$
16. $3\,a + 7 = 19.$
17. $5\,c - 2 = 23.$
18. $4\,t - 7 = 21.$
19. $3\,k = 8 + k.$
20. $6\,x = x + 45.$
21. $7\,y = 3\,y + 12.$
22. $6\,z = 63 - 3\,z.$
23. $10\,r = 80 - 6\,r.$
24. $2\,s = 99 - 9\,s.$
25. $3\,a = 120 - 7\,a.$

26. $15x = 45 - 15x$

27. $4r + 5 = 6 + 3r.$

28. $3x + 9 = 37 - x.$

29. $5t - 8 = 28 + 2t.$

30. $6w - 11 = 88 - 3w.$

31. $15k - 13 = 9k + 17.$

32. $4b + 15 = 35 - b.$

33. $3c - 6 = c + 14.$

34. $-2x + 5 = -15.$

35. $-5y + 9 = -21.$

36. $-11z + 10 = 4 - 8z$

37. $16r - 1 = 4r + 5.$

38. $17t - 7 = 1 - 7t.$

39. $11m + 6 = -9m + 18.$

40. $13p - 9 = -2p + 36.$

41. $19r + 11 = 7r + 13.$

43. In order to solve problems, it is necessary to translate the statements which give the conditions of the problem into algebraic symbols.

EXAMPLE. One number exceeds another number by 18 The product of the smaller number and 3 equals the larger number. Find the numbers.

SOLUTION: 1. Let $s =$ the smaller number
2. Then $s + 18 =$ the larger number,
3. and $3s =$ the product of the smaller and 3.
4. $\therefore 3s = s + 18$, since the product must equal the larger number.
5. S_s: $2s = 18.$
6. D_2: $s = 9.$

CHECK: The smaller is 9; the larger 27; and $3 \times 9 = 27$.

NOTE. The symbol, $\therefore$, means "therefore."

Rule. — **To solve a problem by means of an equation:**

1. Represent one of the unknown numbers by some letter.

2. Represent the other unknown numbers by means of this same letter, using relations given in the problem.

3. From the conditions of the problem form an equation between the numbers; solve the equation.

4. Check the result by comparison with the statements of the problem.

EXERCISE 24

1. What number increased by 11 equals 19?

2. There are two numbers of which the larger is 5 times the smaller. The difference between the numbers is 24. Find the numbers.

3. One number exceeds another by 54. The larger number is 7 times the smaller. Find the numbers.

4. Five times a certain number exceeds 8 by 37. Find the number.

5. If 12 times a certain number is diminished by 3, the result is the same as if 4 times the number is increased by 5. Find the number.

6. 15 exceeds twice a certain number by the same amount that 3 times the number exceeds 10. Find the number.

7. The age of John is double that of his brother James. What are their ages if equal results are obtained by subtracting 5 years from John's age, and adding 10 to James' age.

8 The age of A is twice that of B, and the age of C equals the sum of the ages of A and B. The sum of the ages of A and C exceeds the age of B by 40 years. Find their ages.

9. One angle is four times as large as a second angle; if their sum is increased by 5°, the result is one straight angle. (See § 13) Find the angles.

10. A farmer wishes to inclose a rectangular field for a pasture, making it 15 rods wide. He wants to make it as long as possible, using 172 rods of wire fencing, which he has on hand. How long can he make it?

11. If six times the area of Lake Superior, the largest fresh-water lake, be decreased by 12,000 square miles, the result equals the area of the Caspian Sea, the largest salt-water lake, 180,000 square miles. Find the area of Lake Superior.

12. If twice the height of Mt. McKinley, the highest mountain in North America, be decreased by 11,798 feet, the result equals the height of Mt. Everest, the highest mountain in Asia, 29,002 feet. Find the height of Mt. McKinley.

13. The longest river in the world is the combined Mississippi-Missouri and the next longest is the Nile. Twice the length of the Nile diminished by 2800 miles equals the length of the Mississippi; the length of the Mississippi is 700 miles more than that of the Nile. Find the length of each river.

14. The highest velocity of wind recorded in the United States up to January 1, 1910 exceeded 25 times the lowest average velocity at any point in the United States by 2 miles per hour; the highest velocity exceeds the lowest by 98 miles. Find the highest and lowest velocity.

15. In the United States, the lowest average annual precipitation, rain and snow, is at Yuma, Arizona; and the highest is at Mobile, Alabama. The precipitation in inches at Mobile is 20 times that at Yuma; the sum of the two precipitations is 65.1 inches. Find the precipitation at each place.

44. Percentage and Interest Problems. Many of the problems involving percentage and interest may be expressed and solved by algebraic methods.

EXERCISE 25

Percentage Problems

1. What does 4 % mean? 5 %? r %? m %?

2. What is 4 % of 500? 6 % of 250?

3. Express decimally 4 % of p; 6 % of b; 15 % of c.

4. If the cost of an article is c dollars, and the rate of gain is 25 %, what is the gain? what is the selling price? ($c + .25\,c = ?$.)

5. Find the cost of an article sold for $165 if the gain is 10 %. ($c + .10\,c = 165$.)

6. A grocer wishes to make 25 % on some canned goods. At what price must he buy them so as to be able to sell the goods at $1.25 per dozen?

7. A man wishes to sell hats at $3.50 each. At what price must he buy them so as to make 12 % upon the cost?

8. A real estate agent knows that he can sell a certain lot for $3270. At what price must he buy the lot from the present owner in order that he may make a profit of 9 %?

Interest Problems

9. What is the simple interest on $200 at 6 % for 1 year? for 2 years? for t years?

10. What is the simple interest on R dollars at 6 % for 1 year? for 2 years? for t years?

11. What is the simple interest on $200 at r % for 1 year? for 4 years? for t years?

12. What is the simple interest on P dollars at r % for 1 year? for 3 years? for t years?

13. If I represents the number of dollars interest on P dollars invested at r % for t years, it may be expressed by the formula: $I = \frac{Prt}{100}$. Express this formula in words.

Solve the following problems by substituting in this formula:

14. If a man receives $1150 income from $3500 which has been invested for 4 years and 6 months, what rate of interest has he received?

Solution: 1. Let $r =$ the rate per cent.

2. $P = 3500;\ I = 1150;\ t = 4\frac{1}{2}.$

3. $\therefore 1150 = \overset{35}{\cancel{3500}} \cdot \frac{r}{\cancel{100}} \cdot 4\frac{1}{2}.$ (Since $I = P \cdot \frac{r}{100} \cdot t.$)

$\therefore 1150 = \frac{9}{2} \cdot 35 \cdot r.$

$1150 = \frac{315}{2} r.$

4. M_2: $2300 = 315\, r.$

5 D_{315}: $r = 7.3+ \%.$

15. What principal must be invested at 4 % to yield an income of $ 1500 per year?

16. For what length of time must $ 4000 be invested at 5 % simple interest to yield $ 750 interest?

17. What rate of simple interest has been earned on an investment of $ 2500, if the income is $ 1000 in 10 years?

Amount in Interest Problems

18. The sum of the principal and interest is the amount. Indicate the amount at the end of one year if P dollars are invested at 4 %. ($P + .04\,P = 1.04\,P$.)

Similarly at 6 %; at 7 %.

19 What is the amount at the end of two years if P dollars are invested at 4 %? for 3 years at 5 %?

20. What sum of money will amount to $ 3500 if invested at 5 % simple interest for 5 years?

21. How long will it take $ 1500 to amount to $ 2000 if invested at 5 %? ($2000 = 1500 + 1500 \cdot \frac{5}{100} \cdot t$. Solve the equation for t.)

22. How long will it take $ 1200 to double itself at 6 %?

23. Letting A represent the number of dollars in the amount, and P, r, and t the usual numbers, show that the amount may be expressed by the formula:

$$A = P + \frac{Prt}{100}.$$

Solve the following problems by substituting in this formula:

24. In how many years will $ 3500 amount to $ 4550 at 5 % simple interest?

25. The members of a certain company paid $ 300 per share for some stock. At the end of 7 years, they received $ 800 per share for their stock. What rate of simple interest did their money earn for them during that time?

Historical Note —The symbol % was used first about the year 1685.

120"

$6\frac{2}{3}$ yrs. $04

IV. PARENTHESES

45. Terms which are parts of a single number expression are often inclosed in **Symbols of Grouping.**

The **Parentheses,** ( ), are symbols of grouping; thus,

$$3a-(2x+y-z)$$

means that $2x+y-z$ is to be subtracted from $3a$.

Other symbols of grouping are the **Brackets** [ ], the **Braces,** { }, and the **Vinculum,** ———.

All are used in the same manner as the parentheses, and are referred to collectively as *parentheses.*

Historical Note. — The parentheses are used most commonly now. They were introduced by Girard, a Dutch mathematician, about 1629. Previously the brackets and the braces had been used by Vieta, about 1593, although Bombelli, an Italian, had made the first start in the direction of their use in 1572. Descartes used the vinculum.

REMOVAL OF PARENTHESES

46. Parentheses preceded by a Plus Sign.

Development. 1. Consider $a^4+a^2+(a^3-a)$.

This means that a^3-a is to be added to a^4+a^2.

$$\begin{array}{lcccc} & a^4 & & +a^2 & \\ & & a^3 & & -a \\ \hline \textit{Sum:} & a^4 & +a^3 & +a^2 & -a \end{array} \qquad \begin{aligned} \therefore\ & a^4+a^2+(a^3-a) \\ &= a^4+a^2+a^3-a. \end{aligned}$$

2. Consider $a-b+(c-d)$.

Solution:

$$\begin{array}{lcccc} & a & -b & & \\ & & & +c & -d \\ \hline \textit{Sum:} & a & -b & +c & -d \end{array} \qquad \begin{aligned} \therefore\ & a-b+(c-d) \\ &= a-b+c-d. \end{aligned}$$

3. Notice that the signs of the terms which are within the parentheses are not changed when the parentheses are removed. This fact suggests the

Rule. — To remove parentheses preceded by a plus sign:

Rewrite all terms which are within the parentheses without changing their signs.

47. Parentheses preceded by a Minus Sign.

DEVELOPMENT. 1. Consider $a^4 + a^2 - (a^3 - a)$.

SOLUTION: This means that $(a^3 - a)$ is to be subtracted from $a^4 + a^2$.

$$\begin{array}{rl} & a^4 \qquad + a^2 \\ & \underline{\quad + a^3 \qquad - a} \\ \textit{Remainder:} & a^4 - a^3 + a^2 + a \end{array} \qquad \begin{array}{l} \therefore\ a^4 + a^2 - (+ a^3 - a) \\ = a^4 + a^2 - a^3 + a. \end{array}$$

2. Consider $a - b - (+ c - d)$.

SOLUTION:

$$\begin{array}{rl} & a - b \\ & \underline{\qquad + c - d} \\ \textit{Remainder:} & a - b - c + d \end{array} \qquad \begin{array}{l} \therefore\ a - b - (+ c - d) \\ = a - b - c + d. \end{array}$$

3. Notice that the signs of the terms which are within the parentheses are changed when the parentheses are removed. This fact suggests the

Rule. — To remove parentheses preceded by a minus sign:

Rewrite all terms which are within the parentheses, but change their signs from + to −, or from − to +.

EXAMPLE 1. Remove the parentheses from

$$2a - 3b - (5a - 4b) + (4a - b).$$

SOLUTION: By the rules, the expression becomes

$$2a - 3b - 5a + 4b + 4a - b.$$

When this is simplified, the result is a.

CHECK: Let $a = 1$, and $b = 2$.

$$\begin{aligned} 2a - 3b - (5a - 4b) + (4a - b) &= 2 - 6 - (5 - 8) + (4 - 2) \\ &= 2 - 6 - (-3) + 2 \\ &= 2 - 6 + 3 + 2 = +1. \end{aligned}$$

Also the result $a = 1$.

48. The above rules apply equally to the removal of all symbols of grouping (§ 45).

It should be noticed in each case that the sign of the first term within the symbol of grouping is not the sign prefixed to the symbol of grouping; thus, in $-(a-b)$ the term a is positive within the parentheses.

49. Parentheses sometimes inclose others; in this case, the following rule should be observed by beginners.

Rule. — To remove two or more parentheses, when one incloses another:

1. Combine the terms within the innermost parentheses, if possible; then remove these parentheses according to the rules in §§ 46 and 47.

2. Combine the terms within the resulting innermost parentheses, and remove these parentheses.

3. Continue doing this until all parentheses are removed.

EXAMPLE 2. Simplify $4x-\{3x+(-2x-[x-a])\}$.

The brackets are the innermost symbols of grouping; they are preceded by a minus sign. Remember that the sign of x within the brackets is $+$.

SOLUTION:

1. $4x-\{3x+(-2x-[x-a])\}$
2. $=4x-\{3x+(-2x-x+a)\}$ Removing the [ ].
3. $=4x-\{3x+(-3x+a)\}$ Combining within the ( ).
4. $=4x-\{3x-3x+a\}$ Removing the ( ).
5. $=4x-a.$ Combining within the { }.

EXERCISE 26

Simplify the following expressions by removing the symbols of grouping:

1. $3m-(2n+p)$.
2. $4x+(z-2y)$.
3. $5a-(2b-c)$.
4. $7v+(-3w+x)$.
5. $6r-[-3a+b]$.
6. $8s+\{-3m-t\}$.

7. $9k-[p-2q]$.
8. $2a+(3b-c)$.
9. $l-\{m-n\}$.
10. $8-[-2a+b]$.

In the following examples, since neither symbol of grouping incloses the other, remove both at once. Tell what each example means; thus in 11: "subtract the number $(c-d)$ from the number $(2a-b)$."

11. $(2a-b)-(c-d)$.
12. $(a-b)-(c+d)$.
13. $[r+s]+[t-v]$.
14. $\{r-s\}-\{t+v\}$.
15. $(x-y)-(z+w)$.
16. $(x-y)-(z-w)$.
17. $[m+n]+[p-q]$.
18. $\{m+n\}-\{p-q\}$
19. $(-a+b)+(-c+d)$.
20. $(-a+b)-(-c-d)$.

In the following examples, combine terms after removing the parentheses:

21. $3c-(5c-6)$
22. $(2a-b)-(5a-b)$.
23. $2a-(3a-b)+2b$.
24 $a^2+a-(a^2-a)$.
25. $(5x-6)-(2x-4)$.
26 $(3b-11)-(4b+5)$.
27. $(+x+y)-(-2x+y)$.
28. $5x^2-(-2x^2+x)+6$.
29. $(2r-s)-(3r-2s)$
30. $(3p-2q)+(-3p-2q)$.
31 $9m-(4m+6n)+(3m-n)$.
32 $8r+(5r-[2s+t])$.
33. $3x-(2y+[5x-y])$.
34. $5a-(4a-\{3a-1\})$.
35. $6b-\{7b-(9b+4)-7\}$.
36. $(7t-r)-[8t-(10r+t)-3r]$.
37. $3x-(5x-[7x+9a-4]-3a)$.
38. $5ab-[(3ab-10)-(4ab+7)]$.
39. $7c-(5c+[12c-\{6c+2\}])$.
40. $m-[(6m-7n)-n]-[3m+4n-(2m-3n)]$.

INTRODUCTION OF PARENTHESES

50. Sometimes it is necessary to introduce parentheses into an expression.

Development. 1. What is the rule for removing parentheses preceded by a plus sign?

2. What, then, should be done with the signs of terms which are *placed within* parentheses preceded by a plus sign?

3. Inclose the last two terms of $a + b + c - d$ in parentheses preceded by a plus sign.

4. What is the rule for removing parentheses preceded by a minus sign?

5. What, then, should be done with the signs of terms which are *placed within* parentheses preceded by a minus sign?

6. Inclose the last two terms of $a + b + c - d$ in parentheses preceded by a minus sign.

Rule. — 1. To inclose terms in parentheses preceded by a plus sign, rewrite the terms without changing their signs.

2. To inclose terms in parentheses preceded by a minus sign, rewrite the terms, changing the signs from + to −, and from − to +.

Example. Inclose the last three terms of $r + s - t + v$ in parentheses preceded by a minus sign.

Solution: 1. $r + s - t + v = r - (-s + t - v)$. (Rule 2).

Check: If the parentheses are removed, the result is the original expression.

EXERCISE 27

Inclose the last three terms of the following expressions in parentheses preceded by a plus sign:

1. $m + n - c + d$.
2. $a + b - r - s$.
3. $x + y + z - w$.
4. $r - s + t + x$.
5. $p - q - a + b$.
6. $a^2 + b^2 - 2bc + c^2$.
7. $a^2 - b^2 + 2bc - c^2$.
8. $a^2 - 4b^2 + 12b - 9$.
9. $x^2 - (2yz + y^2 + z^2$.
10. $n^4 - 5n^3 - (8n^2 + 6n + 7$.

11–20. Inclose the last two terms of the foregoing expressions in parentheses preceded by a minus sign.

21–30. Inclose the last three terms in parentheses preceded by a minus sign.

EXERCISE 28

Indicate and simplify, where possible, the following:

1. The sum of a and 5; a and b.

2. The sum of x and $(x+3)$; x and $(x-5)$

3. The difference between a and 5; a and b.

4. The difference between $(a-2)$ and $3a$; $(a+7)$ and $3a$.

5. The sum of $(3a+5)$ and $(2a-6)$; also their difference.

6. The amount by which 15 is greater than 12; greater than a; greater than $(a+2)$.

7. The amount by which $5x$ is greater than 10; greater than $2x$; greater than $(x-4)$.

8. The amount by which 20 exceeds 15; exceeds t; exceeds $(t-3)$.

9. The amount by which $3p$ exceeds 5; exceeds $(p+1)$; exceeds $(p-3)$.

10. The amount by which $(2a+3)$ exceeds $(a-5)$.

11. The smaller part of 15, if l is the larger part.

12. The smaller part of x, if 7 is the larger part.

13. The larger part of n, if 3 is the smaller part.

14. The smaller part of $2a$, if $(a-3)$ is the larger part.

15. The larger part of $(3c-2)$, if $(c+1)$ is the smaller part.

16. The sum of two numbers is 50; the smaller is p. Represent the larger number

17. The sum of two numbers is 37; the larger is l. Represent the smaller.

18. The difference between two numbers is 5; the smaller is s. Represent the larger.

19. The difference between two numbers is x; the smaller is 3. Represent the larger.

20. The difference between two numbers is 10; the larger is n. Represent the smaller.

21. The integer which is consecutive to the integer a.

HINT: 6 is consecutive to 5; 7 is consecutive to 6.

22. The second integer consecutive to a.

23. The sum of a and the consecutive integer.

24. The sum of n and the two consecutive integers

25. If a is an odd number, what is the consecutive odd number?

HINT: 5 is consecutive to 3; 11 to 9; 17 to 15.

51. Parentheses are used in equations and problems.

EXAMPLE. The sum of two numbers is 88. The larger number exceeds the smaller by 36. Find the numbers.

SOLUTION: 1. Let $l =$ the larger number
2. Then $(88 - l) =$ the smaller number.
3. Then $l - (88 - l) = 36.$
4. Removing ( ), $l - 88 + l = 36.$ (§ 47)
$2l - 88 = 36.$
5. A_{88}:* $2l = 124.$
6. D_2: $l = 62,$ the larger number.

$88 - 62 = 26,$ the smaller number.

CHECK: $62 - 26 = 36$

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* See § 42 for the symbol A_{88}.

EXERCISE 29

(Allied.)

Solve the following equations

1. $2r-(r+6)=11.$ $r-6=$
2. $(3x+4)-(2x+9)=15.$
3. $12A+(2A-3)=6A-(-7A-15).$
4. $(4t-5)-(2t+5)=(2t-9)-(4t-7).$
5. $(-10x+12)-(6x-5)=(13-8x)-(15+9x).$
6. $(3x+5)-(2x-7)=(4x+9)-(2x-11).$
7. $(6y-8)-(9y+4)=(9-13y)-11y.$
8. $(13-4m)-(3m+9)=(2m-8)-(6+7m).$
9. $(8t-11)-(7-5t)=12-(13+4t).$
10. $(17z-1)-(9z-10)=15-(13z+6)-2z.$

11. The sum of two numbers is 30. The greater number exceeds the smaller number by 4. Find the numbers.

12. The sum of the ages of A and B is 115 years. A is 13 years younger than B. What are their ages?

13. Divide the number 123 into two parts such that the greater exceeds the smaller by 67.

14. The sum of the ages of A and B is 102 years. A is 26 years older than B. Find their ages.

15. Divide \$93 between A and B so that A may receive \$23 less than B.

16. The Library of Congress at Washington consists of three main stories, whose total height is 64 feet. The height of the second story exceeds that of the first by 7 feet, and is 8 feet less than that of the third story. Find the height of each of the three stories

17. The total length of the Upper Steel Arch Bridge at Niagara Falls is 1240 feet. Its main span, the longest of the kind in the world, exceeds twice the remaining part of the bridge by 40 feet. Find the length of its main span.

18. The first appropriation for a library for Congress was made in 1800. The present Library of Congress was completed at a total expense of over $6,000,000. The cost of the gold leaf on the dome of the present building increased by $1200 equals the appropriation made in 1800; and the sum of that first appropriation and the cost of the gold leaf is $8800. Find each of these amounts.

19. Seven major planets besides the earth revolve around the sun. The number of planets which are farther from the sun than the earth exceeds by one twice the number which are nearer to the sun than the earth. Find the number of planets nearer to and the number farther from the sun than the earth.

20. The sum of two consecutive integers is 35. Find them. (See Example 21, Exercise 28.)

21. The sum of three consecutive integers is 108. Find the numbers.

22. The sum of four consecutive integers is 218. Find them.

23. The sum of two consecutive odd integers is 196. Find them.

24. Find the integer which is such that when increased by the first consecutive integer, and their sum decreased by the second consecutive integer, the result is 75.

25. There are four angles which make up the total angular magnitude around a point. (See § 13, Example 15). The second angle is 3 times the first; the second exceeds the third by 10°, and exceeds the fourth by 50°. Find the angles. Illustrate with a figure.

26. An angle exceeds its supplement by 30°. Find the angle. (See Example 11, § 13).

27. The complement of a certain angle exceeds the angle itself by 20°. Find the angle. (See § 13.)

28. The sum of the supplement and complement of a certain angle is 120°. Find the angle.

29. The fertility of farm land is maintained by adding fertilizers which contain certain plant foods, such as nitrogen, potash, and phosphoric acid. In 100 pounds of a good corn fertilizer, the amount of phosphoric acid should exceed the amount of nitrogen by 10 pounds, and the amount of potash should equal in weight the sum of the other two Find the number of pounds of each plant food in the mixture.

30. Texas is the largest state in the Union, and Rhode Island is the smallest. The area of Texas exceeds 213 times the area of Rhode Island by 72 square miles. The sum of their areas is 267,144 square miles. Find the area of each state

HISTORICAL NOTE — In Examples 5 and 8, negative roots are found The mathematicians of the 16th century were slow to admit that such roots had any meaning. Cardan called them *numeræ fictæ*. Even such men as Vieta and Harriot, who contributed so much to the growth of algebra, admitted only positive solutions. Girard (1590–1632) and Descartes (1673) were especially instrumental in establishing the fact that negative roots should have the same meaning as positive roots.

V. MULTIPLICATION

52. The Law of Signs for Multiplication of positive and negative numbers (§ 25), may be written:

Rule. — 1. The product of two numbers having like signs is positive.

2. The product of two numbers having unlike signs is negative.

HISTORICAL NOTE. — Until the time of Michael Stifel (1553), little progress was made toward finding the rule for multiplying powers of the same base, owing to the cumbersome notation that had been used previously to denote the various powers. Some men even made a table corresponding to our multiplication table, giving the products of some of the powers. Stifel introduced a better notation, used the word exponent, and gave the rule which we now use.

53. The Law of Exponents for Multiplication

DEVELOPMENT. 1. Review the definitions of exponent, base, and power of a number, given in paragraph 15.

2. What does a^3 mean? x^5? r^7? e^8?

3. Write in exponent form:

(a) $b \cdot b \cdot b \cdot b$; (b) $m \cdot m \cdot m$; (c) $y \cdot y \cdot y \cdot y \cdot y$;
(d) $r \cdot r \cdots r$ (if there are 10 factors),
(e) $x \cdot x \cdots x$ (if there are 8 factors).

4. Find the product of a^3 and a^4.

$$a^3 = a \cdot a \cdot a;\ a^4 = a \cdot a \cdot a \cdot a.$$

$$\therefore a^3 \cdot a^4 = (a \cdot a \cdot a) \times (a \cdot a \cdot a \cdot a) = a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a = a^7.$$

5. Find as in 4, the following products and write down the results as in part (a):

(a) $a^3 \cdot a^4 = a^7$. (c) $r^3 \cdot r^5 = ?$
(b) $x^2 \cdot x^4 = ?$ (d) $m^4 \cdot m^5 = ?$

6. Observing the results in 5, see if you can get the following products mentally; check as in 4:

(a) $m^2 \cdot m^3 = ?$

(b) $m^3 \cdot m^6 = ?$

(c) $x^2 \cdot x^5 = ?$

(d) $y^6 \cdot y^2 = ?$

7. The facts observed lead to the

Rule. — To find the exponent of any number in a product, add the exponents of that number in the multiplicand and multiplier.

EXERCISE 30

Find the indicated products

1. $x^3 \cdot x^9$.
2. $a^4 \cdot a^6$.
3. $m^3 \cdot m^7$.
4. $t^4 \cdot t^7$.
5. $t^9 \cdot t^6$
6. $t^5 \cdot t^3$.
7. $m^{10} \cdot m^2$.
8. $s^{12} \cdot s^3$.
9. $y^8 \cdot y^9$.
10. $2^3 \cdot 2^2$.
11. $3^2 \cdot 3^4$.
12. $5^2 \cdot 5^3$

54. The Commutative Law of Multiplication.

DEVELOPMENT 1. 3 × 4 and 4 × 3 each equal 12.

2. Notice, in step 1, that the same factors occur in each product, but that the order is changed. Does changing the order of the factors change the value of the product?

3. Compare 3 × 4 × 5 and 3 × 5 × 4, by finding their values.

4. Write these same factors in some other order, and compare the product with the products obtained in step 3.

5. Arrange the factors 2, 4, and 5 in three different ways and compare the products.

Rule. — The factors of a product may be arranged in any order without changing the value of the product.

EXERCISE 31

Arrange each of the following sets of factors in two ways and find the products.

1. 3, 7.
2. 2, 5, 6.
3. 3, 4, 6.
4. 5, 2, 4.
5. 6, 5, 3.
6. 6, 4, 2.

MULTIPLICATION OF MONOMIALS

55. Development. 1. Find the product of $7a$ and $-2b$.

Solution: 1. $-2b = (-2)b.$ (See § 31)

2. Then $7a(-2b) = 7\cdot a\cdot(-2)\cdot b$

Then, since the order of the factors may be changed, (§ 54)

3. $7a(-2b) = 7\cdot(-2)\cdot a\cdot b$

$= -14ab.$ (§ 5)

2. Find the product of $-5x$ and $-6x^2y$.

Solution: 1. $(-5x)(-6x^2y) = (-5)\cdot x\cdot(-6)\cdot x^2\cdot y.$

$= (-5)\cdot(-6)\cdot x\cdot x^2\cdot y$

$= +30x^3y.$

3. Find similarly the product of $3mn$ and $2p$.

4. Find similarly the product of $+2a^3b$ and $+5ab$.

5. Give at sight, if possible, the following:

(a) $2rs\cdot 5t.$ (c) $7a\cdot 3a^2.$

(b) $3xy\cdot 2z.$ (d) $4m\cdot 5mx.$

Rule. — To find the product of two monomials:

1. Find the product of the numerical coefficients, using the Law of Signs for Multiplication.

2. Multiply this product by the literal factors, giving to each an exponent equal to its exponent in the multiplicand plus its exponent in the multiplier.

Example 1 Multiply $2a^5$ by $9a^4b^2$.

Solution: $(2a^5)(9a^4b^2) = +(2\times 9)\cdot a^{(5+4)}\cdot b^2$

$= 18a^9b^2.$

Example 2. Multiply $-5x^3yz^5$ by $+6xy^2w$.

Solution: $(-5x^3yz^5)(+6xy^2w) = -(5\times 6)\cdot x^{(3+1)}\cdot y^{(1+2)}\cdot z^5\cdot w$

$= -30x^4y^3z^5w.$

EXERCISE 32

Find the following indicated products:

1. $x^5 \cdot x^9$.
2. $a^2 \cdot a^3 \cdot a^4$.
3. $x^5 \cdot x^3 \cdot x^6$.
4. $y^4 \cdot y^3 \cdot y^5$.
5. $m^8 \cdot m \cdot m^2$.
6. $a^{12} \cdot a^4 \cdot a^3$.
7. $E^9 \cdot E^7 \cdot E$.
8. $a^2b^3 \cdot ab$.
9. $x^4y^2 \cdot xy^3$.
10. $a^3x \cdot ax^3$.
11 $x^2yz \cdot xy^2z$.
12. $r^2s^3t \cdot rst^2$.
13. $ab^2c^3 \cdot ab^2c^3$.
14. $r^2s \cdot rs^2t^3$.
15. $2^5 \cdot 2^3$.
16. $4^5 \cdot 4^8$.
17. $5^6 \cdot 5^4$.
18. $10^3 \cdot 10^2$.
19. $pr^2t \cdot r^3$.
20. $mx^2 \cdot mnx^3$.

Multiply:

21. $7\,a^3$ by $3\,a^2$.
22. $-9\,m^2$ by $8\,m^4$.
23. $-5\,ab$ by $2\,a^2$.
24. $-3\,rs^2$ by $-9\,rs$.
25. $5\,xyz$ by $-11\,xyz$.
26. $-11\,xy^2$ by $9\,x^2y$.
27. $-6\,a^3b$ by $-4\,ab^4$.
28 $-6\,x^2y^3$ by $+12\,xy^2$.
29. $-9\,m^2n^3$ by $7\,m^2n^3$.
30. $-12\,rst$ by $-9\,s^2$.
31. $-6\,a^3b^6$ by $-b^3c^5$.
32. $+9\,ax^2$ by $-9\,ay^2$.
33 $8\,x^2z^7$ by $-8\,y^3z^3$
34. $-7\,AB^2$ by $+12\,AB^3C$.
35. $12\,a^2bc$ by $6\,bcd^2$.
36. $13\,a^2$ by $-7\,a^3b^2c$.
37 $-14\,m^2n$ by $-6\,mn^2$.
38. $-16\,a^3bc^4$ by $+5\,a^4b^5c$.
39. $-3\,x^3y^3z^3$ by $-17\,x^2y^3z$
40. $-4\,mn$ by $+17\,m^2n$.

Find the product of:

41. $3\,a^5$, $5\,a^8$, and $6\,a^2$.
42. $-4\,x^3$, $-9\,y^2$, and $2\,z^3$.
43. x^3y^2, $-y^2z$, and xz^3.
44. $-12\,a^7b^2$, $-b^4c^3$, and $-8\,c^3a^2$.
45. a^5, $-3\,a$, $-5\,a^6$, and $-4\,a^3$.

46. $-7m^2n$, $+8mr^2$, and $-2n^2r$.

47. -3 and $+\frac{5a^2}{3}$.

48. $+5$ and $-\frac{2m^2}{5}$.

49. $+4$ and $-\frac{3n}{2}$.

50. $+12$ and $-\frac{5r}{3}$.

51. -15 and $-\frac{6s}{5}$.

52. -10 and $-\frac{7t}{2}$.

53. -9 and $+\frac{4p}{3}$.

54. $+14$ and $-\frac{3w}{7}$.

55. -18 and $+\frac{6s}{9}$.

56. $+20$ and $-\frac{3}{10}x$.

57. $\frac{1}{2}ab^2$ and $-\frac{3}{5}a^2b$.

58. $-\frac{3}{8}mn$ and $-\frac{2}{3}m^2$.

59. $+.5rs^3$ and $-.3r^2s$.

60. $-.12t^3$ and $-.7t$.

56. Numbers, and relations between them, may be represented by geometrical figures.

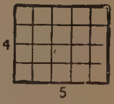


FIG. 1.

EXAMPLE 1 The product 4 × 5 may be represented by the rectangle in Fig. 1. Note that the area of the rectangle is 20 square units.

EXAMPLE 2 The product $4(3+7)$ may be represented by the Fig. 2

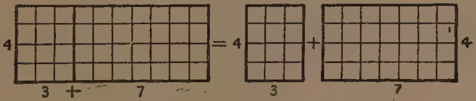


FIG. 2.

$$4(3+7) = 4 \times 3 + 4 \times 7.$$
$$4 \times 10 = \quad 12 \quad + \quad 28.$$

EXAMPLE 3 The product $a(b+c)$ may be represented by the Fig. 3

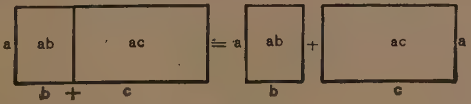


FIG. 3.

$$a(b+c) = ab + ac.$$

EXERCISE 33

1. Draw a figure representing $5(3+4)$, as in Example 2.
2. Draw a figure representing $4(2+3+5)$, as in Example 2.
3. Draw a figure representing $a(b+c+d)$, as in Example 3.

MULTIPLICATION OF POLYNOMIALS BY MONOMIALS

57. Development. 1. From the geometrical illustration in § 56, it is clear that:

$$(a)\ 5(7+3)=(5\times 7)+(5\times 3)=35+15=50.$$

$$(b)\ 6(4+5)=(6\times 4)+(6\times 5)=24+30=54.$$

$$(c)\ a(b+c+d)=ab+ac+ad.$$

In every case, each term of the polynomial is multiplied by the monomial.

Rule. — To multiply a polynomial by a monomial:

1. Multiply each term of the polynomial by the monomial.

2. Unite the results with the proper signs.

Example. Multiply $3a^2-2ab+b^2$ by $-3ab$.

Solution: $(3a^2-2ab+b^2)\times(-3ab)=-9a^3b+6a^2b^2-3ab^3$.

Check: This result is true for any values of a and b. Let $a=1$, and $b=1$.

$3a^2-2ab+b^2=3-2+1=2$, $-3ab=-3$, and $2\cdot(-3)=-6$;

also, $-9a^3b+6a^2b^2-3ab^3=-9+6-3=-12+6=-6$.

EXERCISE 34

Multiply:

1. $4a-9$ by $5a$.
2. m^2-mn+n^2 by mn
3. $3x^2+x-5$ by $-9x^2$.
4. $8x^3y-5xy^3$ by $-3xy^4$.
5. $2a^3-6a^2-7$ by $-7a^3$.
6. $r^2-2rs+s^2$ by $-r^3s^3$.
7. $6x^5-5x^6-7x^4$ by $-7x^3$.
8. $-3c^2-d^2+5cd$ by $4c^2d^3$.
9. $-3x^2y+x^3-3xy^2$ by $-x^2y$.
10. x^5-x^3+x by $-x^3$.

11. $5m^2 - 6mn - 4n^2$
$\underline{3m^2 \qquad\qquad\qquad}$

12. $6p^2 - 5pq + 9q^2$
$\underline{4pq \qquad\qquad\qquad}$

13. $x^2 - 5xy + y^2$
$\underline{-2xy \qquad\qquad}$

14. $a^3 - 3a^2b + 3ab^2 - b^3$
$\underline{-ab \qquad\qquad\qquad\qquad}$

Perform the following indicated multiplications:

15. $7x \cdot (x - 5)$.
16. $-6ab \cdot (10a^2 - 7b^2)$.
17. $x^2y \cdot (x^4 - 4x^2y^2 + y^4)$.
18. $-rs \cdot (r^2 - rs + s^2)$.
19. $-3ab \cdot (a^2 - 2ab + b^2)$.
20. $8x^5 \cdot (6x^3 - 5x - 12)$.
21. $-4a^3b^3 \cdot (3a^2 - 2ab - 4b^2)$.
22. $-5m^4 \cdot (8m^4 - m^2 - 3)$.

23. Simplify the expression: $3(2x - 5) - 2(x + 6)$.

This means that $(2x - 5)$ is to be multiplied by 3; that $(x + 6)$ is to be multiplied by 2; and that the second result is to be subtracted from the first.

SOLUTION:
$$\begin{aligned} 3(2x - 5) - 2(x + 6) &= (6x - 15) - (2x + 12). \\ &= 6x - 15 - 2x - 12. \\ &= 4x - 27. \end{aligned}$$

In the following examples, first tell what each means, as in Example 23; then, simplify.

24. $5(6a + 3b) + 4(5a - 2b)$.
25. $2a(3x - y) - 3a(2x + y)$.
26. $x(x - y) - y(x + y)$.
27. $3(2m + 8) - 2(6 - 5m)$.
28. $3c(2m - 4) - 6c(2m + 4)$.
29. $r(r^2 - s) + s(r - s^2)$.

Multiply and then simplify the following:

30. $6\left(\frac{x}{2} - \frac{x}{3} + \frac{1}{6}\right)$.
31. $12\left(\frac{m}{3} - \frac{2m}{3} + \frac{m}{4}\right)$.
32. $15\left(\frac{r}{5} - \frac{1}{3} + r\right)$.
33. $16\left(\frac{s}{4} - \frac{3s}{2} + \frac{1}{8}\right)$. +2
34. $\frac{1}{2}(2x^3 - 4x + 6)$.
35. $6(\frac{1}{2}x^2 - \frac{1}{6}xy + \frac{1}{3}y^2)$.
36. $10\left(\frac{3t}{5} - \frac{t}{2} + \frac{7}{10}\right)$.
37. $15\left(\frac{3x}{5} - \frac{2}{3} + \frac{x}{3}\right)$.
38. $20\left(\frac{x}{4} - \frac{2x}{5} + \frac{1}{10}\right)$.
39. $24\left(\frac{m}{3} - \frac{3m}{4} + \frac{m}{8}\right)$.
40. $-\frac{1}{3}x^2(3x^4 - 24x^2 + 15)$.
41. $\frac{1}{2}mn(\frac{1}{4}m^2 - \frac{1}{6}mn + \frac{1}{9}n^2)$.

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

58. A number may be multiplied by $(2+3)$ by multiplying first by 2 and then by 3, and adding the products. Thus,

$$(2+3)\times 6=(2\times 6)+(3\times 6)=12+18=30, \text{ for } (5)\times 6=30.$$

Similarly, $$(2+3)\cdot(a+b)=2(a+b)+3(a+b)$$
$$=2a+2b+3a+3b=5a+5b,$$
for $$(5)\cdot(a+b)=5a+5b.$$

The multiplier in each case consists of the sum of 2 and 3; the multiplicand is multiplied separately by 2 and by 3, and the products are added. This illustrates the

Rule. — To multiply one polynomial by another:

1. **Multiply the multiplicand by each term of the multiplier.**
2. **Add the partial products.**

EXAMPLE 1. Multiply $3a-4b$ by $2a-5b$.

SOLUTION: In accordance with the rule, multiply $3a-4b$ by $2a$ and then by $-5b$, and add the partial products. A convenient arrangement is suggested by the arrangement of multiplication problems in arithmetic.

$$\begin{array}{rl}
43 & \\
\underline{12} & \\
86 & =\ 2\times 43 \\
\underline{430} & =10\times 43 \\
516 & =\text{sum}
\end{array}
\qquad\qquad
\begin{array}{lll}
3a & -\ 4b & \\
2a & -\ 5b & \\ \hline
6a^2 & -\ 8ab & =2a(3a-4b) \\
 & -15ab+20b^2 & =-5b(3a-4b) \\ \hline
6a^2 & -23ab+20b^2 & =\text{sum}
\end{array}$$

NOTE: In arithmetic, the multiplication proceeds from right to left; in algebra, the multiplication proceeds from left to right.

EXAMPLE 2. Multiply $a^3-8x^3-2a^2x$ by $2x+a$.

SOLUTION: It is convenient to arrange the multiplicand and multiplier in the same order of powers of some letter (§ 37) and to write the partial products in the same order. Leave spaces for any powers which may not be present in the multiplicand.

Arranging the expressions according to the descending powers of a, we have,

$$\begin{array}{lllll}
a^3 & -2a^2x & & -8x^3 & \\
a & +2x & & & \\ \hline
a^4 & -2a^3x & & -8ax^3 & \\
 & +2a^3x & -4a^2x^2 & & -16x^4 \\ \hline
a^4 & & -4a^2x^2 & -8ax^3 & -16x^4
\end{array}$$

Check. This result should be true for all values of a and x.

Let $a = 1$ and $x = 1$.

Then, $a^3 - 2a^2x - 8x^3 = 1 - 2 - 8 = -9$,

$a + 2x = 1 + 2 = +3$,

and $(-9) \times (+3) = -27$;

also, $a^4 - 4a^2x^2 - 8ax^3 - 16x^4 = 1 - 4 - 8 - 16 = -27$.

EXERCISE 35

Multiply:

1. $x + 3$ by $x + 5$.
2. $r - 7$ by $r - 4$.
3. $2s - 5$ by $s - 3$.
4. $3m + 2$ by $m - 4$.
5. $4t - 9$ by $t + 3$.
6. $3x + 7$ by $2x + 3$.
7. $2m + 5$ by $5m - 1$.
8. $6p - 3$ by $2p + 7$.
9. $5y - 1$ by $6y - 8$.
10. $7z + 10$ by $4z - 5$.
11. $2a + b$ by $a + b$.
12. $3c - 2d$ by $2c + d$.
13. $5r + 6s$ by $3r - 2s$.
14. $5x - 2y$ by $3x - 4y$.
15. $6m - 3p$ by $4m + 5p$.
16. $7y - 9z$ by $6y + 8z$.
17. $11a + 5d$ by $6a - 4d$.
18. $12p + 7q$ by $8p - 7q$.
19. $2x^2 - y^2$ by $x^2 - 3y^2$.
20. $9w^2 - 7v$ by $11v^2 + 3v$.
21. $m^2 - m - 3$ by $m + 3$.
22. $2a^2 + 7a - 9$ by $5a - 1$.
23. $x^2 - 2xy + 3y^2$ by $x - 3y$.
24. $x^2 - xy + y^2$ by $x + y$.
25. $x^2 + 4xy + 16y^2$ by $x - 4y$.
26. $m^2 + mn + n^2$ by $m^2 - mn + n^2$.
27. $a^3 + a - 2 - 2a^2$ by $a^2 + 2a - 3$.
28. $3 + a^3 - 7a - 4a^2$ by $2a + 1$.
29. $9x + 2x^2 - 5$ by $4 + 3x^2 - 7x$.
30. $6n - 8 + 4n^2$ by $-4 + 2n^2 - 3n$.
31. $9r^2 - 5y^2 + 6ry$ by $8ry + 4y^2 + 7r^2$.
32. $3a^2 - 5ab - 8b^2$ by $4a^2 - 9ab - 7b^2$.

33. $a-b+c$ by $a-b-c$.

34. $r+s+t$ by $r-s-t$.

35. $2n^2+m^2+3mn$ by $2n^2-3mn+m^2$.

36. $a^3+3ab^2-3a^2b-b^3$ by a^2+b^2-2ab.

37. $m^4-3m^3+9m^2-27m+81$ by $m+3$.

38. $4a+6b+10c$ by $2a-3b+5c$.

39. $x^4+4x^2+8x+2x^3+16$ by $x-2$.

40. $a^2+b^2+c^2+ab-bc+ac$ by $a-b-c$.

41. $\frac{1}{3}a-\frac{1}{4}b$ by $\frac{1}{3}a+\frac{1}{4}b$.

42 $\frac{1}{2}m-\frac{1}{3}n$ by $\frac{1}{2}m+\frac{1}{3}n$.

43. $\frac{1}{4}x-\frac{1}{9}y$ by $\frac{1}{4}x+\frac{1}{9}y$.

44. $2a-\frac{1}{3}b$ by $a+\frac{1}{2}b$.

45. $\frac{2}{3}x-\frac{3}{4}y$ by $\frac{3}{2}x-\frac{4}{3}y$.

46. $(3x-5)^2$.

47. $(2m-3n)^3$.

48. $(4r+5)^3$.

49. $(2x-3)^4$.

50. $(3a-4b)^3$.

Find the product of the following:

51. $a+3$, $a-4$, and $a+2$.

52. $m+4$, $2m-3$, and $m-5$.

53. $x+y$, x^2-xy+y^2, and x^3-y^3.

54. $m+n$, m^2+n^2, and $m-n$.

55. x^2+xy+y^2, x^2-xy+y^2, and x^2-y^2.

PARENTHESES IN MULTIPLICATION

59. Example. Simplify $(a-2x)^2-2(3a+x)(a-x)$.

Solution: To simplify this expression, first multiply $(a-2x)$ by itself (§ 15); second, find the product of 2, $3a+x$ and $a-x$; third, subtract the second result from the first.

1.
$$\begin{array}{r}
a-2x \\
a-2x \\
\hline
a^2-2ax \\
-2ax+4x^2 \\
\hline
a^2-4ax+4x^2
\end{array}$$

2.
$$\begin{array}{r}
3a+x \\
a-x \\
\hline
3a^2+\ ax \\
-3ax-x^2 \\
\hline
3a^2-2ax-x^2 \\
2 \\
\hline
6a^2-4ax-2x^2
\end{array}$$

3. **Then** $(a-2x)^2-2(3a+x)(a-x)$
4. $=(a^2-4ax+4x^2)-(6a^2-4ax-2x^2)$
5. $=a^2-4ax+4x^2-6a^2+4ax+2x^2$ (§ 47)
6. $=-5a^2+6x^2$. *Answer.*

Note. Be careful to place the results of steps 1 and 2 in parentheses as in step 4.

EXERCISE 36

Tell, as in the above solution, what must be done to simplify the following; then simplify:

1. $(3a+8)(a-6)+(2a+7)(4a-9)$.
2. $(2m+7)(3m-5)-(2m-5)(3m+7)$.
3. $(a-2x)(a+3x)+(a+2x)(a-3x)$.
4. $(2a-3b)^2-4(a-b)(a+5b)$.
5. $2(h+3)(h-2)-(h+5)(h-6)$.
6. $5(x-4)(x+1)-3(x-3)(x+2)$.
7. $2(3x+2)(4x-3)-(3x-2)(4x+3)$.
8. $3(3a+5)(2a-8)-2(4a-7)(a+6)$.
9. $4(3x-2)(x+6)-5(2x-7)(x+2)$.
10. $(a+b)(a^2+b^2)-(a-b)(a^2-b^2)$.

USE OF MULTIPLICATION IN EQUATIONS

60. Example 1. Seven times the complement of a certain angle exceeds twice its supplement by 20°. Find the angle.

Solution: 1. Let $a=$ the number of degrees in the angle.
2. Then, $90-a=$ the number of degrees in the complement.
3. and, $180-a=$ the number of degrees in the supplement.
4. Hence, $7(90-a)=2(180-a)+20$.
5. Multiplying, $630-7a=360-2a+20$.
6. Combining, $630-7a=380-2a$.
7. S_{630}: $-7a=-250-2a$.
8. A_{2a}: $-5a=-250$.
9. M_{-1}: $5a=250$.
10. D_5: $a=50$.

Check: The angle is one of 50°. The complement contains 40° and the supplement, 130°. Does $7\times 40°=2\times 130°+20°$? Yes.

EXAMPLE 2. Solve the equation,

$$(2a+5)(3a-7)-(2a-5)(3a+7)=4.$$

SOLUTION: 1. $(2a+5)(3a-7)-(2a-5)(3a+7)=4.$

2. Multiplying, $(6a^2+a-35)-(6a^2-a-35)=4.$

3. Removing ( ), $6a^2+a-35-6a^2+a+35=4.$

4. Combining terms, $2a=4.$

5. D_2: $a=2.$

CHECK: Does $(2\cdot 2+5)(3\cdot 2-7)-(2\cdot 2-5)(3\cdot 2+7)=4$?

Does $(9)\cdot(-1)-(-1)(+13)=4$?

Does $-9-(-13)=4$?

Does $-9+13=4$? Yes.

NOTE: In Exercises like Example 2, be careful to put the products obtained in step 1, in parentheses as in step 2.

EXERCISE 37

Solve the following equations:

1. $2(m-3)=20.$ **2.** $4(y-5)+7=15.$

3. $3(2x-4)+2(x-5)=5(x+1).$

4. $6(2-3x)+3=3(4x-5).$

5. $12-5(3a-2)=2(a-6).$

6. $2(v+9)+3(v-4)=16.$

7. $4(t-3)+3(2t+5)=33(4-t).$

8. $10-5(3l-4)=6(3-2l).$

9 $7g-6(2g-5)=6(6-g).$

10. $3n-2(2n-7)=3(n-2).$

11. $3(4m-5)-4(m-6)=3(m+17)-7.$

12. $(x-5)(x+6)-(x+3)(x-4)=0.$

13. $(y-7)(y+2)-(y-9)(y+3)=0.$

14. $(2r+3)(3r-5)-6(r-4)(r-3)+5=0.$

15. $3(2s-4)(s+7)-2(3s-2)(s+5)=-3(s-2).$

16. The sum of two numbers is 75. The larger exceeds the smaller by 11. Find the numbers.

17. The sum of two numbers is 100. If four times the greater be diminished by 22, the result is 5 times the smaller. Find the two numbers.

18. The distance from New York to Paris exceeds the distance from New York to London by 280 miles. Four times the distance to London exceeds three times the distance to Paris by 2900 miles. Find the two distances.

19. One number exceeds another number by 7. If 6 times the smaller is diminished by 5 times the larger, the remainder is 5. Find the numbers.

20. Separate 60 into two parts such that 4 times the smaller shall exceed 2 times the larger by 30.

21. The sum of two numbers is 80. If twice the greater be decreased by 12, the result exceeds 4 times the smaller by 4. Find the numbers.

22 The Library of Congress stands upon a rectangular base whose perimeter is 1620 feet. The length exceeds the width by 130 feet. Find the dimensions of the building.

23. There are two consecutive numbers such that the sum of twice the smaller and three times the larger is 78. What are the numbers?

24. There are two consecutive integers whose product exceeds the square of the larger by 20. What are they?

25. The total population of Chicago, Philadelphia and Greater New York (1910 Census), was 8,501,174. The population of Chicago exceeded the population of Philadelphia by 636,275; the population of New York exceeded twice the population of Chicago by 396,317. Find the population of each of the cities.

EXERCISE 38

Algebraic Expression

1. A is now 15 years of age. Express his age:

(*a*) 5 years ago; (*b*) m years ago; (*c*) y years ago;
(*d*) 8 years from now; (*e*) m years from now.

2. B is b years of age. Express his age:

(*a*) 4 years from now; (*b*) m years from now;
(*c*) 6 years ago; (*d*) x years ago; (*e*) t years ago.

3. A is now x years of age. B's present age exceeds the age of A by 5 years.

(*a*) Express B's present age; (*b*) the sum of their ages.
(*c*) Express the age of each 10 years ago.
(*d*) Express the age of each 10 years from now.

4. A is now a years of age; B is twice as old.

(*a*) Express B's present age.
(*b*) Express the age of each 3 years ago.
(*c*) Express the age of each 7 years from now.
(*d*) Express the fact that B's age 5 years ago was 3 times A's age at that time.

5. Express the value of:

(*a*) $(15-x)$ pounds of tea at 40 ¢ per pound;
(*b*) x pounds of tea at 60 ¢ per pound;
(*c*) the entire amount of tea.

6. Express the value in cents of:

(*a*) x nickels; (*b*) $2x$ dimes; (*c*) $3x$ dollars;
(*d*) $4x$ quarters; (*e*) all of the coins.

7. Express the value in cents of

(*a*) d dimes; (*b*) $(15-d)$ quarters;
(*c*) $(15+3d)$ half dollars; (*d*) all of the coins.

8. Express in inches:

(a) m feet; (b) $3m$ yards.

(c) the combined length of m feet and $3m$ yards.

9. Express in pints:

(a) $3x$ pints plus $2x$ quarts plus $5x$ gallons.

(b) $2c$ pints plus $(3c-2)$ quarts plus $(5-3c)$ gallons.

EXERCISE 39

1. The sum of the ages of A and B is 50 years; in 5 years A will be 5 times as old as B. Find their ages.

Solution: 1. Let a = the number of years in A's age *now*.

2. Then $(50-a)$ = the number of years in B's age *now*.

3. Then $a+5$ = the number of years in A's age in 5 yr.

4. and $(50-a+5)$ or $55-a$ = the number of years in B's age in 5 yr.

5. $\therefore (a+5)=5(55-a)$.

Complete the solution.

Note. Represent with care the present ages of both persons; also their ages at the other time mentioned; then form the equation.

2. A father is now 9 times as old as his son. In 9 years he will be only 3 times as old as his son. What are their present ages?

3. The difference between the present ages of a father and son is 25 years. In 10 years the father will be twice as old as his son. What are their present ages?

4. A is 5 times as old as B. In 9 years he will be only 3 times as old as B. What are their ages?

5. B is twice as old as A. 35 years ago he was 7 times as old as A. What are their present ages?

6. A is 68 years of age, and B is 11. In how many years will A be 4 times as old as B?

Hint. Let N equal the number of years. Find the age of each in N years, and then form the equation.

7. A is 25 years of age and B is 65. How many years ago was B 6 times as old as A?

8. A grocer has two grades of tea, a 60¢ grade and a 90¢ grade. He wishes to make a mixture which he can sell for 80¢ per pound. How many pounds of each must he use in a mixture of 120 pounds?

Solution: 1. Let $n =$ the number of pounds of 60¢ tea used.
2. $\therefore (120 - n) =$ the number of pounds of 90¢ tea used.
3. $\therefore 60\,n =$ the value of the 60¢ tea in cents.
4. and $90(120 - n) =$ the value of the 90¢ tea in cents
5. $\therefore 60\,n + 90(120 - n) =$ the value of the mixture in cents
6. But $120 \times 80 =$ the value of the mixture in cents.
7. $\therefore 60\,n + 90(120 - n) = 9600.$
8. From the equation $n = 40$, $120 - n = 80$.

Check: 40 pounds of tea at 60¢ are valued at $24.
80 pounds of tea at 90¢ are valued at $72.
Total value of the mixture is $96.
Also, 120 pounds at 80¢ are valued at $96.

9. A grocer has tea worth 70¢ per pound and other tea worth 40¢ per pound. How many pounds of each must he take to form a mixture of 50 pounds which he may sell at 49¢ per pound?

10. A grocer has coffee which he sells at 36¢ per pound, and other coffee which he sells at 20¢ per pound. How many pounds of each must he take to make a mixture of 100 pounds which he may sell at 25¢ per pound?

11. A seedsman wishes to make a mixture of grass seed consisting of clover seed and blue grass seed. He sells his clover seed at 40¢ per pound, and his blue grass seed at 22¢ per pound How many pounds of each must he take to make a mixture of 200 pounds which he may sell for 25¢ per pound?

12. A sum of money amounting to $2.80 consists of dimes and quarters. The number of dimes exceeds the number of quarters by 7. Find the number of each kind of coin.

45 44 45

Solution: 1. Let $q =$ the number of quarters.
2. Then $q + 7 =$ the number of dimes.
3. $\therefore 25q =$ the number of cents in the quarters,
and $10(q + 7) =$ the number of cents in the dimes.
4. $\therefore 25q + 10(q + 7) = 280.$
Complete the solution and check it.

13. A man has two kinds of money, dimes and fifty-cent pieces. If he is offered \$4.00 for 20 coins, how many of each kind must he give?

14. A sum of money amounting to \$2.20 consists of five-cent pieces and quarters. There are in all 16 coins. How many are there of each kind?

15. A sum of money amounting to \$24.90 consists of \$2 bills, fifty-cent pieces and dimes. There are 5 more fifty-cent pieces than \$2 bills, and 3 times as many dimes as \$2 bills. How many are there of each denomination?

61. Equations having Fractional Coefficients.

Example 1. If the sum of a certain number and one half of itself be diminished by three fifths of the number, the remainder is 9. Find the number.

Solution: 1 Let $x =$ the number.

Then $x + \frac{x}{2} - \frac{3x}{5} = 9.$

2. The denominators must be eliminated.

M_{10}: $10\left(x + \frac{x}{2} - \frac{3x}{5}\right) = 10 \cdot 9.$ (§ 10)

3. $10x + \overset{5}{\cancel{10}}\,\frac{x}{\cancel{2}} - \overset{2}{\cancel{10}} \cdot \frac{3x}{\cancel{5}} = 90.$ (§ 57)

4. $10x + 5x - 6x = 90.$

5. Combining, $9x = 90.$

6. D_9 $x = 10.$ (§ 9)

Check: Does $10 + \frac{\overset{5}{\cancel{10}}}{\cancel{2}} - \frac{3 \cdot \overset{2}{\cancel{10}}}{\cancel{5}} = 9$?

$10 + 5 - 6 = 9$? Yes.

Note. In order to eliminate the denominators, multiply the equation by the Lowest Common Multiple of the denominators.

Example 2. Solve the equation $\frac{7m}{6} - \frac{5}{3} = \frac{3m}{5} - \frac{1}{4}$.

Solution: 1. The L. C. M. of 6, 3, 5, and 4 is 60.

2. M_{60}: $$60\left(\frac{7m}{6} - \frac{5}{3}\right) = 60\left(\frac{3m}{5} - \frac{1}{4}\right). \quad (\S\ 10)$$

3. $$70m - 100 = 36m - 15.$$

4. A_{100}: S_{36m}: $$70m - 36m = 100 - 15. \quad (\S\ 41)$$

5. $$\therefore 34m = 85;\ m = \tfrac{85}{34} = \tfrac{5}{2} = 2.5.$$

Check: This solution may be checked by substitution or by going over the solution again. If the latter method is used, great care must be taken, as it is easy to overlook an error.

EXERCISE 40

Solve the following equations:

1. $\frac{x}{2} - \frac{x}{3} = \frac{5}{6}$.
2. $\frac{m}{3} - \frac{m}{5} = \frac{8}{15}$.
3. $r + \frac{r}{3} = \frac{8}{3}$.
4. $\frac{2s}{3} = \frac{s}{4} + 5$.
5. $\frac{5a}{3} = \frac{3a}{4} + \frac{11}{6}$.
6. $\frac{m}{2} + 1 = \frac{3m}{5}$.
7. $\frac{2t}{7} = \frac{t}{3} + 1$.
8. $\frac{6w}{5} = 1 + \frac{5w}{4}$.
9. $\frac{5x}{14} = \frac{4x}{7} - \frac{3}{2}$.
10. $\frac{y}{3} + \frac{4y}{8} = \frac{5}{2}$.
11. $\frac{r}{2} + \frac{2r}{3} - \frac{5r}{6} = 6$.
12. $m - \frac{3m}{5} = 9 - \frac{m}{2}$.
13. $\frac{c}{2} - \frac{5}{3} = \frac{3c}{4} - \frac{2c}{3}$.
14. $4d - \frac{6d}{5} = \frac{d}{2} + \frac{23}{20}$.
15. $\frac{4t}{9} - \frac{5t}{6} = \frac{2}{3} - \frac{3t}{2}$.
16. $a + \frac{a}{2} = 11 - \frac{a}{3}$.
17. $\frac{2z}{3} - \frac{9}{8} = \frac{5z}{6} - \frac{5}{4}$.
18. $\frac{3y}{2} - \frac{y}{3} = \frac{5y}{4} - \frac{1}{8}$.
19. $\frac{4t}{9} - 1 = \frac{7t}{9} - \frac{16}{3}$.
20. $\frac{7m}{2} - \frac{4m}{3} = \frac{11}{6} - \frac{2m}{5}$.

EXERCISE 41

1. One fifth of a certain number exceeds one eighth of the same number by 3. Find the number.

2. The sum of three numbers is 65. The second is one half of the first, and the third is two thirds of the first. Find the numbers.

3. What number increased by one half of itself equals the sum of two thirds of itself and 25?

4. What number exceeds the sum of its third, sixth, and fourteenth parts by 18?

5. What number is such that if four sevenths of it be subtracted from itself, the result equals the excess of three fourths of the number over 18?

6. What number is such that if two thirds of it be increased by 100, the result equals four fifths of it?

7. Seven eighths of a certain number is as much less than 21 as three tenths of it exceeds $2\frac{1}{2}$. What is the number?

8. The difference between the third and fifteenth parts of a certain number is 28. Find the number.

9. In a triangle commonly used by draughtsmen, the second angle is two thirds of the first, and the third angle is one half of the second. Find the angles of the triangle. (§ 13)

10. In another triangle used by draughtsmen, there are two equal angles, each of which is one half of the third angle. Find the angles of this triangle.

11. There are three consecutive numbers such that the sum of the second and third exceeds three halves of the first by 9. Find the three numbers.

12. A man has $4.35 in dollars, dimes, and cents. He has one fourth as many dollars as dimes, and five times as many cents as dollars. How many coins of each kind does he have?

13. The Treasury at Washington is one of the most imposing of the national buildings. Its perimeter is 1400 feet. Its width exceeds one half of its length by 25 feet. Find its dimensions.

14. The greatest depth of Lake Superior is one half that of Lake Michigan; the greatest depth of Lake Huron exceeds one sixth that of Lake Michigan by 700 feet. The depth of Lake Huron exceeds that of Lake Superior by 100 feet. Find the depth of each.

15. Probably the largest room in the world under one roof is the passenger concourse of the Union Station in Washington, D.C. Its perimeter is 1780 feet. One fifth of its length exceeds its width by 22 feet. Find its dimensions.

16. Ten times the population of the United States in 1820, in millions, exceeded the population in 1910 by 3.8 millions; the population in 1910 exceeded 7 times the population in 1820 by 25 millions. Find the population in both years.

17. Plants feed upon certain plant foods present in the soil, such as potash, nitrogen, and phosphoric acid. A fair crop of potatoes will remove from an acre of ground about 99 pounds of these three foods. The amount of potash removed is 5 times, and the amount of nitrogen $2\frac{1}{4}$ times that of phosphoric acid. Find the number of pounds of each removed.

18. The length of the foundation of the Capitol in Washington exceeds twice the width by $51\frac{1}{3}$ feet. The perimeter of the foundation is $2202\frac{2}{3}$ feet. Find the dimensions of the foundations of the Capitol.

19. The average wholesale value of oak lumber in 1899 was $13.78 per thousand feet. This exceeded one half of the wholesale value in 1909 by $3.53. What was the wholesale value per thousand in 1909?

20. The distance from San Francisco to London *via* New York is 6990 miles. The part of the journey by rail is 50 miles less than $\frac{15}{17}$ of the part by water. Find the part of the journey on land and the part on water.

VI. DIVISION

62. Division is the process of finding one of two numbers when their product and the other number are given.

To divide 15 by 3 means to find the number by which 3 must be multiplied to give the product 15.

The **Dividend** is the product of the numbers; it is the number divided.

The **Divisor** is the other given number; it is the number by which the dividend is divided.

The **Quotient** is the required number.

63. It is clear that $a \div a = 1$; for $a \times 1 = a$.

64. It is agreed that the product of zero and any number is zero. This makes division by zero impossible.

Thus, if we try to find the quotient of $6 \div 0$, and let q equal the quotient, we should have the relation

$$6 = 0 \cdot q.$$

But $0 \cdot q = 0$ and not 6, so there cannot be any ordinary number to use as q. Hence, there is no number to represent the quotient of $6 \div 0$

65. Division is **indicated** by writing a fraction whose numerator is the dividend and whose denominator is the divisor.

Thus, the quotient of $15 \div 5$ is written $\frac{15}{5}$.

The quotient of $7\,abc \div 3\,xy$ is written $\dfrac{7\,abc}{3\,xy}$.

NOTE. The line, —, was used to indicate division long before the symbol, $\div$.

66. Division of a Product by a Number.

EXAMPLE. Divide 6×8 by 2.

SOLUTION: 1. $6 \times 8 \div 2 = \dfrac{6 \times 8}{2}$.

2. If 6 is divided by 2, $\frac{6\times 8}{2}=\frac{\overset{3}{\cancel{6}}\times 8}{\cancel{2}}=24.$

3. If 8 is divided by 2, $\frac{6\times 8}{2}=\frac{6\times \overset{4}{\cancel{8}}}{\cancel{2}}=24.$

4. If both 6 and 8 are divided by 2,

$$\frac{6\times 8}{2}=\frac{\overset{3}{\cancel{6}}\times \overset{4}{\cancel{8}}}{\cancel{2}}=12.$$

5. Since we know that $\frac{6\times 8}{2}=48\div 2$ or 24, it is clear that the results obtained in steps 2 and 3 are correct, but that the result in step 4 is incorrect. Hence,

Rule. — To divide the product of two or more numbers by a number, divide any one of the factors by the number, but divide only one of them by it.

EXERCISE 42

Find each of the following indicated divisions in two ways:

1. $\frac{9\times 12}{3}$. 2. $\frac{18\times 24}{6}$. 3. $\frac{28\times 56}{7}$.

67. The Law of Signs for Division.

Since $(+2)\times(+3)=+6$, then $(+6)\div(+2)=+3$.
Since $(-2)\times(+3)=-6$, then $(-6)\div(-2)=+3$.
Since $(+2)\times(-3)=-6$, then $(-6)\div(+2)=-3$.
Since $(-2)\times(-3)=+6$, then $(+6)\div(-2)=-3$.

If the signs of the dividend, the divisor, and the quotient in each of the previous statements are examined, the following rules become clear:

Rule. — 1. The quotient of two numbers having like signs is positive.

2. The quotient of two numbers having unlike signs is negative.

EXERCISE 43

1. Divide each of the following numbers by $+3$:
$+12$; $+15$; $+27$; -18; -36; -42; -57.

2. Divide each of the following numbers by $+2$:
-18; $+48$; $+72$; -24; -96; $+54$; -108.

3. Divide each of the numbers in Example 2 by:
(*a*) -3; (*b*) $+6$; (*c*) -2; (*d*) -12.

4. Divide each of the numbers in Example 1 by -3.

5. Divide each of the numbers in Example 1 by -2.

68. The Law of Exponents for Division.

DEVELOPMENT. 1. Review the definitions of *exponent, base,* and *power* of a number in § 15.

2. Divide a^5 by a^3.

SOLUTION:
$$\frac{a^5}{a^3}=\frac{\overset{1}{\cancel{a}}\cdot\overset{1}{\cancel{a}}\cdot\overset{1}{\cancel{a}}\cdot a\cdot a}{\underset{1}{\cancel{a}}\cdot\underset{1}{\cancel{a}}\cdot\underset{1}{\cancel{a}}}=1\cdot a\cdot a=a^2.$$

Therefore $a^5 \div a^3 = a^2$. CHECK: $a^3 \cdot a^2 = a^5$.

Each a in the denominator is divided into one of the a's in the numerator. The quotient in each case is 1, since $a \div a = 1$.

3. Find as in step 2 the following quotients and write the results as in part a:

(*a*) $a^5 \div a^3 = a^2$.
(*b*) $y^6 \div y^4 = ?$
(*c*) $m^7 \div m^4 = ?$
(*d*) $t^{10} \div t^6 = ?$

4. Examine carefully the exponents in the dividend, the divisor, and the quotient. In the following problems, try to give the results immediately without going through the solution as in step 2. Test by multiplication.

(*a*) $p^8 \div p^5 = ?$
(*b*) $a^9 \div a^7 = ?$
(*c*) $b^{10} \div b^6 = ?$
(*d*) $c^8 \div c^3 = ?$

5. Divide a^4b^6 by a^3b^2.

SOLUTION: $a^4b^6 \div a^3b^2 = \dfrac{a^4b^6}{a^3b^2} = \dfrac{\overset{1}{\cancel{a}}\cdot\overset{1}{\cancel{a}}\cdot\overset{1}{\cancel{a}}\cdot a\cdot\overset{1}{\cancel{b}}\cdot\overset{1}{\cancel{b}}\cdot b\cdot b\cdot b\cdot b}{\underset{1}{\cancel{a}}\cdot\underset{1}{\cancel{a}}\cdot\underset{1}{\cancel{a}}\cdot\underset{1}{\cancel{b}}\cdot\underset{1}{\cancel{b}}} = ab^4.$

Rule. — The exponent of any number in the quotient is equal to its exponent in the dividend minus its exponent in the divisor.

HISTORICAL NOTE. — This rule was known to Stifel (see note § 53).

DIVISION OF MONOMIALS BY MONOMIALS

69. EXAMPLE 1. Divide $-14\,a^3b^2$ by $+7\,a^2$.

SOLUTION: Use the Law of Signs, § 67, and the Law of Exponents, § 68.

$$\frac{-14\,a^3b^2}{+7\,a^2} = -2\,a^{(3-2)}b^2 = -2\,ab^2.$$

CHECK: $(+7\,a^2)(-2\,ab^2) = -14\,a^3b^2.$

EXAMPLE 2. Divide $54\,a^7b^2c^3$ by $-9\,a^4b^2c^2$.

SOLUTION: $\dfrac{54\,a^7b^2c^3}{-9\,a^4b^2c^2} = -6\,a^{(7-4)}b^{(2-2)}c^{(3-2)} = -6\,a^3b^0c = -6\,a^3c.$

CHECK: $(-6\,a^3c)\times(-9\,a^4b^2c^2) = +54\,a^7b^2c^3.$

Notice that by the law of exponents, $b^2 \div b^2 = b^{2-2} = b^0$. No meaning has been given to the zero power of a number. Since $b^2 \div b^2$ must equal 1, we agree that $b^0 = 1$.

The zero power of any number is 1. Thus:

$$a^0 = 1;\ 5^0 = 1;\ c^0 = 1.$$

Rule. — To divide a monomial by a monomial:

1. Make the quotient positive, if the dividend and divisor have like signs; make it negative, if they have unlike signs.

2. Find the quotient of the absolute values of the numerical coefficients.

3. Multiply the quotient of step 2 by the product of the literal factors, giving each its exponent in the dividend minus its exponent in the divisor.

4. Omit any literal factor which has the same exponent in the dividend and divisor.

Example. Divide $-33\,a^6bx^2y^4$ by $+3\,a^5x^2y$.

Solution. $(-33\,a^6bx^2y^4) \div (+3\,a^5x^2y) = -11\,aby^3$. *Ans.*

Check: These solutions may be checked by substitution, for they must be correct for all values of the literal numbers (except 0 sometimes). A better way is to use the rule that the divisor times the quotient equals the dividend.

Here, does $(+3\,a^5x^2y) \times (-11\,aby^3) = -33\,a^6bx^2y^4$? Yes.

EXERCISE 44

Divide:

1. x^5 by x^3.
2. r^7 by r^4.
3. p^9 by p^3.
4. m^4 by m^2.
5. a^2b^3 by ab.
6. r^3s^2 by r^2s.
7. x^5y^4 by x^2y^3.
8. $a^{10}b^2$ by a^4b.
9. $12\,a^4$ by $2\,a^3$.
10. $15\,x^5$ by $3\,x^2$.
11. $20\,r^3s^2$ by $4\,r^2s^2$.
12. $18\,c^3d^5$ by $3\,cd^3$.
13. $-14\,mn$ by $-2\,m$.
14. $+16\,a^2b$ by $-4\,ab$.
15. $-10\,xy^2$ by $+5\,xy$.
16. $-21\,p^3$ by $-7\,p^2$
17. $-18\,x^3y$ by $2\,xy$.
18. $2\,m^5n^3$ by m^3n^2.
19. $-36\,a^6b^4c^2$ by $+6\,a^4b^3c^2$
20. $-96\,x^7y^{11}$ by $-16\,x^3y^6$
21. $-24\,a^4b^2c$ by $-8\,a^4b^2$.
22. $28\,x^5yz^3$ by $-7\,x^3y$.
23. $-33\,a^6x^2y^4$ by $-3\,a^4y^2$.
24. $65\,x^6y^5z^3$ by $-13\,xy^5$.
25. $28\,a^7b^6c^8$ by $-14\,a^2bc^7$.
26. $-72\,x^7y^8$ by $-6\,x^7y^8$.
27. $-40\,a^2b^4c^5$ by $-8\,bc$.
28. $-55\,x^5y^3z^6$ by $-11\,y^2z^6$.
29. $-70\,a^7b^3c^2$ *by* $14\,ab^3c$.
30. $-96\,m^3n^4$ by $-12\,mn^4$.
31. $64\,ab^2c$ by $-4\,ab^2c$.
32. $-63\,r^4s^6$ by $7\,r^2s^3$.
33. $3\,mn$ by $-6\,m$.
34. $-5\,r^2s$ by $15\,r^2$
35. $4\,x^2y$ by $-\frac{1}{2}\,xy$.
36. $12\,ab^2c$ by $-24\,ab^2$.
37. 10^5 by 10^3.
38. 2^8 by 2^5.
39. $3^5 \cdot 4^2$ by $3^4 \cdot 4^2$.
40. $7^3 \cdot 8^2 \cdot 5$ by $7^2 \cdot 5$.

DIVISION OF POLYNOMIALS BY MONOMIALS

70. Development. 1. Since $2 \times 9 = 18$, then $\frac{18}{2} = 9$.

2. Since $2(x+3) = 2x + 6$, then $\dfrac{2x+6}{2} = x + 3$.

3. Since $3(a-5) = 3a - 15$, what does $\dfrac{3a-15}{3}$ equal?

4. What does each of the following equal? Test the result by multiplying the divisor and quotient.

(a) $\dfrac{2x+2}{2}$; (b) $\dfrac{6r+4}{2}$; (c) $\dfrac{6m-9}{3}$.

5. Since $a(b+c) = ab + ac$,

then $\dfrac{ab+ac}{a} = b + c$.

Rule. — To divide a polynomial by a monomial:

1. Divide each term of the dividend by the divisor.

2. Unite the results with their proper signs.

Example 1. Divide $12x^3 - 6x^2 + 3x$ by $-3x$.

Solution: $\dfrac{12x^3 - 6x^2 + 3x}{-3x} = -4x^2 + 2x - 1.$

Check: $(-4x^2 + 2x - 1) \cdot (-3x) = 12x^3 - 6x^2 + 3x.$

Example 2. Divide $-9a^3 + 3a^2 - 12a^4$ by $-3a^2$.

Solution: $(-12a^4 - 9a^3 + 3a^2) \div (-3a^2) = 4a^2 + 3a - 1.$

Check: Multiply the quotient by the divisor; the result should equal the dividend.

EXERCISE 45

Divide:

1. $3a - 6b$ by 3.
2. $16r - 8s$ by 8.
3. $12x^2 - 16y^3$ by 4.
4. $20a^2 - 15a$ by $5a$.
5. $x^3 - x^2 + x$ by x.
6. $-3x^4 + 6x^2$ by $-3x^2$.
7. $+21r^3 - 14r^2s$ by $-7r^2$.
8. $18m^3n - 27mn^3$ by $9mn$.
9. $-44a^2b + 55a^3b$ by $11a^2b$.
10. $36c^3d^3 - 48c^2d^4$ by $12c^2d^3$.

11. $16x^9 + 28x^6 - 24x^3$ by $4x^2$.

12. $104m^2n^3 - 52m^3n^2$ by $-13mn$.

13. $6a^3b^7c^3 - 15a^6b^3c^5 + 3a^4b^8c$ by $-3a^3b^2c$.

14. $-63x^5y^6z^2 - 84x^3y^4z^7$ by $+7x^3yz^2$.

15. $20m^8n^5 - 35m^4n^3 - 30m^4n^9$ by $-5m^4n^3$.

16. $-36a^{11} + 108a^9 - 60a^7$ by $-12a^5$.

17. $32a^8b^2c - 24ab^3c^2 + 48ab^2c^3$ by $-8abc$.

18. $-63x^{10} - 18x^7 + 45x^6 - 99x^4$ by $-9x^4$.

19. $-12x^4y + 6x^3y^2 - 16x^2y^3 + 20xy^4$ by $-2xy$.

20. $60a^{14} - 30a^{12} + 15a^{10} - 45a^8$ by $-15a^7$.

21. $(a^4b - a^3b^2 + a^2b^3) \div (a^2b)$.

22. $(8x^4 - 12x^3y) \div (-4x^3)$.

23. $(-21c^3d^2 - 42cd^3) \div (-7cd^2)$.

24. $(-6m^3 + 9m^2 - 12m) \div (-3)$.

25. $(-x^4 + x^2 - 1) \div (-1)$.

DIVISION OF POLYNOMIALS BY POLYNOMIALS

71. Division of a polynomial by a polynomial is like long division in arithmetic.

Divide 864 by 24.

1. $86 \div 24 = 3+$.
2. $24 \times 3 = 72$; subtract . . . .
3. $14 \div 2 = 6+$.
4. $24 \times 6 = 144$; subtract . . . .

$$\begin{array}{r|l} & 36 \\ \hline 24 & 864 \\ & \underline{72} \\ & 144 \\ & \underline{144} \end{array}$$

Divide $10x^3 - 21x^2 - 11x + 12$ by $2x^2 - 3x - 4$.

1. $10x^3 \div 2x^2 = 5x$.
2. $(2x^2 - 3x - 4) \times (5x)$; subtract . . . .
3. $(-6x^2) \div (2x^2) = -3$.
4. $(2x^2 - 3x - 4) \times (-3)$; subtract . . . .

$$\begin{array}{r|r} & 5x - 3 \\ \hline 2x^2 - 3x - 4 & 10x^3 - 21x^2 - 11x + 12 \\ & \underline{10x^3 - 15x^2 - 20x \phantom{{}+12}} \\ & -6x^2 + 9x + 12 \\ & \underline{-6x^2 + 9x + 12} \end{array}$$

The following explanation of the process may be given.

We are to find an expression which, when multiplied by the divisor, $2x^2-3x-4$, will produce the dividend.

The term containing the highest power of x in the product is the product of the terms containing the highest powers of x in the multiplicand and multiplier.

Therefore, $10x^3$ is the product of $2x^2$ and the term containing the highest power of x in the quotient. Dividing $10x^3$ by $2x^2$ gives $5x$, which is the term containing the highest power of x in the quotient.

When the dividend is formed, the divisor is multiplied by each term of the quotient, and the results are added. Now reversing the process, multiply the divisor by $5x$ and subtract the result, $10x^3-15x^2-20x$, from the dividend.

The remainder, $-6x^2+9x+12$, must be the product of the divisor and the rest of the quotient. Consider it a new dividend.

Its term containing the highest power of x, $-6x^2$, is the product of $2x^2$ and the term containing the next lower power of x in the quotient. Dividing $-6x^2$ by $2x^2$ gives the next term, -3 Multiply the divisor by -3 and subtract the result from the previous remainder There is now no remainder.

The quotient is therefore $5x-3$

Rule. — To divide a polynomial by a polynomial:

1. Arrange the dividend and the divisor in either ascending or descending powers of some common letter.

2. Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.

3. Multiply the whole divisor by the first term of the quotient; write the product under the dividend and subtract it from the dividend.

4. Consider the remainder a new dividend, and repeat steps 1, 2, and 3.

NOTE 1. The terms of the quotient are placed above the terms of the dividend from which they are obtained.

NOTE 2. The like terms are carefully arranged in a vertical column

NOTE 3. Spaces should be left for any powers of the common letter which are not present in the dividend

NOTE 4 As in arithmetic, there may be a final remainder

EXAMPLE 1 Divide $9ab^2+a^3-9b^3-5a^2b$ by $3b^2+a^2-2ab$

SOLUTION: 1. Arrange according to descending powers of a.

$$\begin{array}{r|l} & a-3b \\ \hline a^2-2ab+3b^2 & a^3-5a^2b+9ab^2-9b^3 \\ & a^3-2a^2b+3ab^2 \\ \hline & \quad -3a^2b+6ab^2-9b^3 \\ & \quad -3a^2b+6ab^2-9b^3 \\ \hline \end{array}$$

2. $a^3 \div a^2 = a$.
3. $(a^2-2ab+3b^2)\times a$; subtract . . . .
4. $-3a^2b \div a^2 = -3b$.
5. $(a^2-2ab+3b^2)\times(-3b)$; subtract . . . . .

Check: Let $a = 1$ and $b = 1$.

Divisor: $a^2 - 2ab + 3b^2 = 1 - 2 + 3 = 2$.

Dividend: $a^3 - 5a^2b + 9ab^2 - 9b^3 = 1 - 5 + 9 - 9 = -4$.

Dividend ÷ divisor: $(-4) \div (2) = -2$.

Quotient: $a - 3b = 1 - 3 = -2$.

Example 2. Divide $x^2y^2 + x^4 - y^4$ by $-xy + y^2 + x^2$.

Solution:

$$\begin{array}{r|l} & x^2 + xy + y^2 \\ \hline x^2 - xy + y^2 & x^4 \qquad + x^2y^2 \qquad - y^4 \\ & x^4 - x^3y + x^2y^2 \\ \hline & \quad + x^3y \qquad\qquad - y^4 \\ & \quad + x^3y - x^2y^2 + xy^3 \\ \hline & \qquad\qquad x^2y^2 - xy^3 - y^4 \\ & \qquad\qquad x^2y^2 - xy^3 + y^4 \\ \hline & \qquad\qquad\qquad -2y^4 \end{array}$$

Note. The quotient is $x^2 + xy + y^2$; the remainder is $-2y^4$. As in arithmetic, the complete quotient may be written:

Complete quotient: $x^2 + xy + y^2 + \dfrac{-2y^4}{x^2 - xy + y^2}$.

Check: Let $x = 1$, and $y = 1$. Then, dividend $= 1$, and divisor $= 1$.

$$\textit{Quotient} = 1 + 1 + 1 + \frac{-2}{1 - 1 + 1} = 3 + \frac{-2}{1} = 3 - 2 = 1.$$

Since $1 \times 1 = 1$, the quotient is correct.

Another check would be to multiply the divisor by the quotient and add the remainder; the sum should equal the dividend.

EXERCISE 46

Divide:

1. $x^2 + 5x + 6$ by $x + 2$.
2. $x^2 + 7x + 12$ by $x + 4$.
3. $y^2 + 7y + 10$ by $y + 5$.
4. $m^2 + 8m + 12$ by $m + 2$.
5. $A^2 + 11A + 24$ by $A + 3$.
6. $r^2 - 12r + 32$ by $r - 8$

7. $s^2 - 13s + 42$ by $s - 7$.

8. $t^2 + 63 - 16t$ by $t - 9$.

9. $c^2 + 72 - 17c$ by $c - 8$.

10. $d^2 - 12d + 36$ by $d - 6$

11. $x^2 + x - 6$ by $x - 2$.

12. $t^2 - t - 30$ by $t + 5$.

13. $a^2 - 3a - 28$ by $a - 7$.

14. $m^2 + 2m - 15$ by $m + 5$.

15. $n^4 - 7n^2 - 30$ by $n^2 + 3$.

16. $x^2 - 17ax + 60a^2$ by $x - 5a$.

17. $a^2 + 5ab - 66b^2$ by $a + 11b$.

18. $x^2 - 2xz - 35z^2$ by $x - 7z$.

19. $x^4 + 5x^2y - 24y^2$ by $x^2 - 3y$.

20. $x^2y^2 - 15xy + 36$ by $xy - 3$.

21. $15x^2 - 11x - 14$ by $3x + 2$.

22. $6a^2 + 35 - 29a$ by $2a - 5$.

23. $12a^2 - 28a + 15$ by $6a - 5$.

24. $30x^2 + 8 - 53x$ by $6x - 1$.

25. $32x^2 - 15y^2 + 28xy$ by $4x + 5y$.

26. $25m^2 + 40mn + 16n^2$ by $5m + 4n$.

27. $x^3 - 6x^2 - 19x + 84$ by $x - 7$.

28. $6a^3 - 18a - 11a^2 + 20$ by $2a - 5$.

29. $4x^3 - 12y^3 + 17xy^2 - 12x^2y$ by $2x - 3y$.

30. $12a^3 + 6ab^2 + 5b^3 - 23a^2b$ by $4a - 5b$.

31. $x^4 + 4xy^3 + 3y^4$ by $x^2 - 2xy + 3y^2$.

32. $2n^2 - 4 + 5n^3 - 19n$ by $-8n + 5n^2 - 3$.

33. $12 + 13x^2 - 19x - 12x^3$ by $-3x^2 - 4 + x$.

34. $2a^4 + 8a - a^3 + 15$ by $2a^2 - 3a + 5$.

35. $-9m^2 - 16 + m^4 - 24m$ by $3m + m^2 + 4$.

36. $x^4 + y^4 + x^2y^2$ by $x^2 + y^2 - xy$.

37. $x^3 + 8$ by $x + 2$.

38. $x^4 - 16$ by $x - 2$.

39. $x^4 - y^4$ by $x + y$.

40. $x^4 - y^4$ by $x - y$.

41. $x^5 + y^5$ by $x + y$.

42. $x^5 + y^5$ by $x - y$.

43. $x^5 + 32$ by $x + 2$.

44. $1 - 16a^4$ by $1 + 2a$.

45. $n^4 - 16$ by $2n^2 + 8 + 4n + n^3$.

46. $13x^3 + 71x - 70x^2 - 20 + 6x^4$ by $4 + 3x^2 - 7x$.

47. $n^8 + 4m^2n^4 + 16m^4$ by $2mn^2 + 4m^2 + n^4$.

48. $63x^4 + 114x^3 + 49x^2 - 16x + 20$ by $9x^2 - 5 + 6x$.

49. $x^5 + 50 - 70x + 37x^2$ by $10 - 2x + x^2$.

50. $10ab^3 - a^2b^2 - 25b^4 + 16a^4$ by $5b^2 + 4a^2 - ab$.

51. $x^2 + \frac{3}{2}x - 1$ by $x + 2$.

52. $x^2 - \frac{10}{3}x + 1$ by $x - \frac{1}{3}$.

53. $6x^2 - \frac{5}{6}x - \frac{1}{6}$ by $2x - \frac{1}{2}$.

54. $\frac{1}{6}a^2 + \frac{13}{6}a + 6$ by $\frac{1}{3}a + 3$.

Historical Note. — Stifel (1486–1567) seems to have been one of the first to divide a polynomial by a polynomial. Sir Isaac Newton (1642–1727), in a book published in 1707, pointed out the advantage of arranging the dividend and divisor according to ascending or descending powers of the same letter.

VII. SIMPLE EQUATIONS

72. **An Equation** expresses the equality of two numbers. Equations are of two kinds.

73. **An Identity** or **Identical Equation** is an equation whose members are equal for all values of the literal numbers involved; as, $3x(a-b) = 3ax - 3bx$

If $\quad a = 3,\ b = 1,\ x = 2,\ 3x(a-b) = 3 \cdot 2(3-1) = 6 \cdot 2 = 12;$
also, $\quad 3ax - 3bx = 3 \cdot 3 \cdot 2 - 3 \cdot 1 \cdot 2 = 18 - 6 = 12.$

Any other set of values of a, b, and x will produce equal numerical results in the two members of the equation.

An identity is like a declarative sentence; it makes a statement of actual equality.

74 An equation is said to be *satisfied* by a set of values of the letters involved in it when, after substituting these values for the letters, the equation becomes an identity

Thus, $\quad xa - xb = 2a - 2b$ is satisfied by $x = 2$, for
$2a - 2b = 2a - 2b$ is an identity
$x - y = 5$ is satisfied by $x = 8$, $y = 3$, for
$8 - 3 = 5$ is an identity

75 **A Conditional Equation** is an equation involving one or more literal numbers, which is not satisfied by all values of the literal numbers

Thus, (*a*) $x + 2 = 5$ is not satisfied by any value of x except $x = 3$

(*b*) $x^2 - 5x = -6$ is satisfied by $x = 2$ and by $x = 3$, but by no other values of x.

A conditional equation is like an interrogative sentence; it implies a question

Thus, $3x - 5 = 4$, asks "for what value of x is $3x - 5 = 4$?" The answer is, "x must be 3," for $3 \times 3 - 5 = 9 - 5 = 4$.

The word "equation" usually refers to a conditional equation.

76. If an equation contains only one unknown number, any value of the unknown number which satisfies the equation is called a **Root** of the equation.

To solve an equation is to find its root or roots.

Thus, 3 is the root of the equation $x + 2 = 5$.

77. If an equation has only one unknown number, if the unknown does not appear in the denominator of any fraction, and if the unknown appears only with the exponent 1, then the equation is called **an Equation of the First Degree,** or a **Simple Equation.**

Thus, $3x - 5 = 4$ is a simple equation.

HISTORICAL NOTE. — The idea of the *degree* of an equation was introduced by Descartes.

PROPERTIES OF EQUATIONS

78. Previously, in solving equations, four rules have been employed:

1. The same number may be added to both members of an equation without destroying the equality. (§ 41.)
2. The same number may be subtracted from both members of an equation without destroying the equality. (§ 41.)
3. Both members of an equation may be multiplied by the same number without destroying the equality. (§ 10.)
4. Both members of an equation may be divided by the same number without destroying the equality. (§ 9.)

All simple equations are solved by the application of one or more of these rules. However, observation of the results of solving equations by means of these rules leads to certain more mechanical methods of solution which may be used.

79. Transposition. Solve the equation $10x - 5 = 3x + 30$.

SOLUTION: 1. $10x - 5 = 3x + 30$.
2. A_5: $10x = 3x + 30 + 5$.
3. S_{3x}: $10x - 3x = 30 + 5$.
4. $7x = 35$.
5. $x = 5$.

In equation 3, the term $-3x$ in the left member corresponds to the term $+3x$ in the right member of equation 1; and the term $+5$ in the right member of equation 3 corresponds to the term -5 in the left member of equation 1. These are two examples of *transposition*. The result is the same *as if* a term were taken from one member of the equation and placed in the other, with its sign changed

Rule.—A term may be transposed from one member of an equation to the other, provided its sign is changed.

HISTORICAL NOTE.—Our word *algebra*, curiously, is associated with this process, *transposition*. About the first quarter of the ninth century an Arabian mathematician, Mohammed ben Musa, wrote an algebra, for the title of which he used, Ilm al-jabr wa'l muqabalah. Al-jabr meant the process of transposing terms. This title was used in various forms in Europe until about the fifteenth century, when the last part was dropped and *algebra* came into use.

The Greeks had no special name for their algebra. The Hindu writers called it *reckoning with unknowns*.

80 Cancelling Terms in an Equation.

EXAMPLE. Solve the equation $x + a = b + a$.

SOLUTION: $x + a = b + a$.
S_a: $x = b$.

Thus, the term a, which appeared in both members of the given equation, does not appear at all in the next equation; the result is the same *as if* the term a were *simply* dropped from both members.

Rule.—If the same term, preceded by the same sign, occurs in both members of an equation, it may be cancelled.

81. Changing Signs in an Equation.

Example. Solve the equation $a - x = b - c$.

Solution: 1. $a - x = b - c$.

2. M_{-1}: $-a + x = -b + c$.

or $x - a = c - b$.

Thus, the signs of all terms of the equation in step 2 are exactly opposite to the signs of these terms in the equation of step 1. The result is the same *as if* the signs of all the terms of the equation were *simply* changed.

Rule. — The signs of all of the terms of an equation may be changed, without destroying the equality.

Note. The rules given in §§ 79, 80, and 81 are valuable, but the student should endeavor to remember that they arise out of the more fundamental rules given in § 78.

Example. Solve the equation

$$7 - 5x - 9x = 15 - 9x - 3x.$$

Solution: 1. $7 - 5x - 9x = 15 - 9x - 3x.$

2. Cancelling the term $-9x$: $7 - 5x = 15 - 3x.$ (§ 80)

3. Transposing $+7$ and $-3x$:

$$-5x + 3x = 15 - 7. \quad (\S\, 79)$$

$$-2x = 8.$$

4. Changing the signs of the terms: (§ 81)

$$2x = -8.$$

5. D_2: $x = -4.$

Check as usual.

EXERCISE 47

Find the roots of the following equations; check the solution:

1. $5a + 5 = 61 - 3a.$
2. $9m - 7 = 3m - 37.$
3. $13 - 6x = 13x - 6.$
4. $7t + 10 = 16t - 17.$
5. $15 - 6n = 5n + 48.$
6. $13 + 4p = 11p - 22.$
7. $5r - 12 = 16 - 9r.$
8. $21 - 15z = -8z - 7.$
9. $30 + 17c = 27c + 22.$
10. $19 - 16y = 27 - 28y.$

11 $2(5m + 1) + 16 = 4 + 3(m - 7).$

12. $8t - 5(4t + 3) = -3 - 4(2t - 7).$

13 $5c - 6(3 - 4c) = c - 7(4 + c).$

14. $2(4x + 7) - 6(2x + 3) = 8(3x - 4) - 7(2x - 3).$

15. $10r - (3r + 2) = 9r - (5r - 4).$

16. $19 - 5c(4c + 1) = 40 - 10c(2c - 1).$

17. $3 - (x - 3) = 5 - 2x.$

18. $4(m - 7) = 5(m + 10) - 6(m + 8).$

19. $2(r - 1) = 4(r - 5) - 3(r - 2).$

20. $5 = 3(x - 2) - 10(x - 6).$

82 No general rule can be given for the solution of problems. The following suggestions will prove helpful:

1. Every problem gives a relation between some unknown numbers.

2 There are as many distinct statements as there are unknown numbers.

3 Represent one of the unknown numbers by a letter; then, using all but one of the statements, represent the other unknowns in terms of that same letter.

4. Using the remaining statement, form an equation.

EXERCISE 48

1. Divide 44 into two parts such that one divided by the other shall give 2 as the quotient and 5 as the remainder.

HINT: The dividend = divisor × quotient + remainder.

2. If 11 be added to a certain number, and the sum be multiplied by 5, the product equals -6 times the number. Find the number.

3 Divide 19 into two parts such that 7 times the less shall exceed 6 times the greater by 3.

4. Divide 38 into two parts such that twice the greater shall be less by 22 than 5 times the less.

5. The age of a father is 5 times that of his son; his age 5 years from now will exceed 3 times his son's age by 4 years. Find their present ages.

6. There are three consecutive odd integers such that when three times the first is increased by the second, the sum exceeds 3 times the third by 5. Find the numbers.

7. Divide $22 among A, B, and C so that A may receive $2.25 more than B and $1.75 less than C.

8. Divide 49 into two parts such that one divided by the other may give 2 as quotient and 7 as remainder.

9. Twice the width of the Pennsylvania Station in New York exceeds its length by 80 feet. Four times the length exceeds the perimeter by 700 feet. Find the dimensions.

10. Find the three sides of a triangle if the perimeter is 45 inches, if the second side is twice the third side, and if the first side exceeds the third by 5 inches.

11. Divide 134 into two parts such that one divided by the other may give 3 as quotient and 26 as remainder.

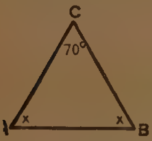
12. The elevation of Mt. Whitney, in California, the highest point recorded in the United States, is 14,501 feet, measured from sea level. The lowest point of dry land in the United States is in Death Valley, California. If 52 times the elevation of Death Valley be diminished by 45 and the result be increased by the elevation of Mt. Whitney, the sum is zero. Find and interpret the elevation of Death Valley

13. A now has one third as much money as B; after B gives him $24, he will have 3 times as much money as B has left. How much has each?

14. A cab driver finds at the end of the day that he has $11.55. He has 3 less nickels than quarters, twice as many half-dollars as quarters, and as many dimes as he has nickels and quarters together. How many of each kind of coin has he?

$p = 3$ $y = 2$ $s = 8$

15. A gardener decides to buy $25 worth of gladiolus bulbs. He wants some of the pink variety which sell at $2 a hundred; two thirds as many of the yellow variety, at $3.50 per hundred, as of the pink variety; and four times as many of the scarlet variety, at $1.50 per hundred, as of the yellow variety. How many of each shall he order?



16. In an isosceles triangle, two sides are equal and, also, the angles opposite these sides are equal. Find the three angles of an isosceles triangle if the angle between the equal sides is 70°.

17. Find the sides of an isosceles triangle if its perimeter is 720 inches and its base is 150 inches.

18. The highest temperature recorded in the United States up to 1907 was 119°, recorded in Arizona. The lowest temperature was recorded at one time in Montana. If twice the lowest temperature be decreased by 9 and the result be added to the highest temperature, the result is zero. Find and interpret the lowest temperature.

19. The area of Nebraska exceeds the area of Virginia by 34,893 square miles; the area of California exceeds three times the area of Virginia by 30,416 square miles; and the area of California exceeds twice the area of Nebraska by 3257 square miles. Find the area of each of the states.

20. In 1910, the total number of boys and girls in the public secondary schools was 915,061. The number of boys exceeded three fourths of the number of girls by 11,123. Find the number of boys and of girls.

21. The total annual income from two investments is $250. One sum is invested at 4 % and the other sum, which exceeds the first by $500, is invested at 5 %. Find each of the sums invested.

Solution: 1. Let $s =$ the smaller sum in dollars.

2. $\therefore .04\,s =$ the interest on this sum.

3. $\therefore (s + 500) =$ the larger sum in dollars.
4. $\therefore .05(s + 500) =$ the interest on this sum.
5. $\therefore .04\,s + .05\,(s + 500) = 250.$
6. $\therefore .04\,s + .05\,s + 25 = 250.$
7. $\therefore .09\,s = 225.$
8. $\therefore s = 2500.$
9. $\therefore s + 500 = 3000.$

Check: 5 % of $ 3000 = $ 150; 4 % of $ 2500 = $ 100;
and $ 150 + $ 100 = $ 250.

22. One sum of money is invested at 5 %; a second sum is invested at 6 %. If 3 times the first sum exceeds the second sum by $ 100, and if the total income is $ 155, find the sums invested.

23. A man has $ 5000 invested at 4 %. How much money must he invest at 6 % to make the total income equivalent to 5 % on the total amount invested?

24. A man has $ 3000 invested at 3.5 %, and $ 4500 at 4 %. How much must he invest at 6 % to make the total income equivalent to 5 % on the total sum invested?

25. A man owns a number of shares of U. S. Steel Preferred Stock ($ 1000 par value) which pay 7 % annually, and 5 times as many bonds of the Chicago Edison Company, ($ 1000 par value) which pay 5 %. If his total income is $ 960, how much has he invested in each form?

83. Distance, Rate, and Time Problems. — If a train goes a distance of 240 miles in 6 hours, it travels at an average rate of 40 miles per hour.

The *time* (t) is expressed as a number of units of time; as hours, minutes, days.

The *rate* (r) is expressed as a number of units of distance covered in the unit of time; as, a number of miles per hour, or a number of feet per second, etc.

The *distance* (d) is expressed as a number of units of distance covered in the total time.

From the example and the definitions, it is clear that:

the distance equals the rate multiplied by the time.

$$d = rt. \tag{1}$$

From equation (1), $\frac{d}{t} = r$ or $r = \frac{d}{t}$; that is

the rate equals the distance divided by the time.

From the equation (1), $\frac{d}{r} = t$ or $t = \frac{d}{r}$; that is,

the time equals the distance divided by the rate;

thus the time occupied in going 200 miles at 40 miles per hour is 5 hours.

EXERCISE 49

1 Express the distance covered by a train in 15 hours at the rate of:

(*a*) 5 miles per hour; (*c*) $(x + 7)$ miles per hour;

(*b*) R miles per hour; (*d*) $(2y - 3)$ miles per hour.

2. Express the distance covered by a train in H hours at the rate of:

(*a*) m miles per hour; (*b*) $(x + 9)$ miles per hour.

3. Express the time required by an automobile to go a distance of 300 miles at the rate of:

(*a*) 30 miles per hour; (*c*) $(x + 5)$ miles per day;

(*b*) n miles per hour; (*d*) $(m - 4)$ miles per day.

4. Express the time for a trip of N miles at the rate of:

(*a*) 10 miles per hour; (*b*) x miles per hour.

5. At what rate does a man travel who goes 250 miles:

(*a*) in 10 days; (*c*) in $(x - 5)$ hours;

(*b*) in n days; (*d*) in $(r + 7)$ days.

6. At what rate does a man travel who goes D miles: (*a*) in 15 hours; (*b*) in t days; (*c*) in $(x-4)$ minutes.

7. The rate of one train is r miles per hour. Express the rate of a train which travels 5 miles more per hour

8. Express the distance traveled by each of the trains in Example 7 in 15 hours.

9. Suppose that the distance gone by the second train exceeds that gone by the first train by 75 miles. Form an equation expressing this fact.

10. A man on foot and a man on a bicycle both travel for 5 hours, the rate of the latter exceeding that of the former by 7 miles per hour. Let r represent the rate of the former

(*a*) Express the rate of the second man.

(*b*) Express the distance each travels.

(*c*) Form an equation expressing the fact that the sum of the distances is 60 miles.

Equations

In the following problems, express the time, rate, and distance traveled by each party, and then form the equation from the given relations. It is usually wise to illustrate the problems geometrically.

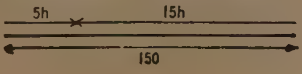
11. Two men travel toward each other from points which are 150 miles apart at rates of 5 and 15 miles an hour respectively. In how many hours will they meet?

SOLUTION: 1. Let h = the number of hours until they meet

2.

Then for	the time is	the rate is	the distance
one man	h hours	5 m. an hr	$5h$ miles
the other man	h hours	15 m. an hr.	$15h$ miles

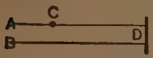
3. Since the sum of the distances is 150 miles, $5h + 15h = 150$



12. Suppose that the more rapid traveler starts two hours after the other in Problem 11. When will they meet?

13. Suppose that two men, who travel at the rate of 6 miles and 10 miles per hour respectively, start from the same place in opposite directions. In how many hours will they be 200 miles apart?

14. Suppose that A, traveling 10 miles per hour, leaves a place 3 hours before B; suppose that B travels 15 miles per hour. In how many hours will B overtake A?



HINT: A is at *C* when B starts; B overtakes A at *D*.

15 Suppose A, traveling 15 miles per hour, starts 4 hours before B. At what rate must B travel to overtake A in 10 hours?

16. Two hours after A left, B starts after him in an automobile at the rate of 27 miles an hour and overtakes him in $2\frac{1}{2}$ hours. At what rate was A traveling?

17. A and B travel toward each other from points separated by 250 miles, A at a rate which exceeds B's rate by 8 miles an hour. If they meet in 5 hours, at what rate did each travel?

18. Some boys who are boating on a river know that they can go with the current 6 miles per hour and can return against the current at the rate of 3 miles per hour. How far may they go if they have only 3 hours for the trip?

19. A man has 11 hours at his disposal. How far may he go in a buggy at the rate of 10 miles an hour if he plans to return at an average rate of 7 miles per hour?

20. An automobile is traveling at the rate of 25 miles an hour. In how many hours will a second automobile overtake the first if the second starts 2 hours later than the first, and travels at the rate of 35 miles an hour?

21. An express train whose rate is 36 miles an hour starts 54 minutes after a slow train and overtakes it in 1 hour and 48 minutes. What is the rate of the slow train?

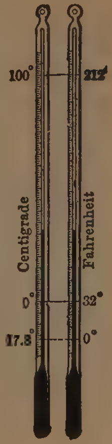
22. An automobile party is traveling at the rate of 20 miles per hour. At what rate must a second automobile travel in order to overtake the first if it starts 2 hours after the first and wishes to overtake it in 3 hours?

23. Chicago, and Madison, Wisconsin are about 140 miles apart. Suppose that a train starts from each city toward the other, one at the rate of 35 miles per hour and the other at the rate of 40 miles per hour. How soon will they meet?

84. Problems about Thermometers. There are two kinds of thermometers in common use, the **Fahrenheit** and the **Centigrade.** The Fahrenheit is the one with which most of us are familiar. The Centigrade is used by scientists throughout the world.

It is necessary at times to change a temperature reading on one scale to the corresponding reading on the other scale.

The temperature at which water boils and that at which it freezes are called the "boiling" and the "freezing" points. On the Fahrenheit scale these points are marked 212° and 32°; on the Centigrade scale, 100° and 0° respectively. The number of Fahrenheit degrees between these two points is 180, and the number of Centigrade is 100. Hence, 100 Centigrade degrees correspond to 180 Fahrenheit degrees, or 1 Centigrade degree to $\frac{9}{5}$ Fahrenheit degree.



NOTE. This does not mean that a temperature of 1° C. is the same as $\frac{9}{5}$° F. 1° C. is one degree above 0; the corresponding Fahrenheit reading is $\frac{9}{5}$° above 32 (the freezing point), or $33\frac{4}{5}$°. Thus, a temperature of 1° C. = a temperature of $33\frac{4}{5}$° F.

EXERCISE 50

1. How many Fahrenheit degrees are equal to the following number of Centigrade degrees?

(*a*) 15; (*b*) 25; (*c*) 50; (*d*) 100.

2. Remembering that Centigrade degrees above freezing are counted from zero, and Fahrenheit from 32, what Fahrenheit temperature corresponds to the following Centigrade temperature?

(*a*) + 15° C.

Solution 1. 15 Centigrade degrees = 27 Fahrenheit degrees.

2. 15° C. above freezing = 27° F. above 32 = 59° F.

∴ 15° C corresponds to 59° F.

(*b*) + 30° C.; (*c*) + 55° C.; (*d*) − 10° C.

3. Derive a formula for changing Centigrade temperature readings into Fahrenheit readings.

Solution: 1. Let $C°$ = the Centigrade reading.
Let $F°$ = the Fahrenheit reading.

2. C Centigrade degrees $= (\frac{9}{5}C)$ Fahrenheit degrees.

3. C counted from 0, the Centigrade freezing point, $= (\frac{9}{5}C)°$ counted from 32, the Fahrenheit freezing point.

$$\therefore F = 32 + \tfrac{9}{5}C.$$

Check: Let $C = 0$. $\therefore F = 32 + \frac{9}{5} \cdot 0 = 32$.
Let $C = 100$. $\therefore F = 32 + \frac{9}{5} \cdot 100 = 32 + 180 = 212$.

Since the freezing and boiling temperatures correspond, the solution is correct.

4. The formula can be used to change Fahrenheit into Centigrade readings.

Change − 13° F. to Centigrade.

Solution 1. $-13 = 32 + \frac{9}{5}C$. (Substituting in the formula.)

2. $\therefore -65 = 160 + 9C$.

3. $\therefore -225 = 9C$, or $C = -25°$;
i.e. 25° below zero Centigrade.

5. In Physics and Chemistry, the temperature − 273° C. is important. To what Fahrenheit temperature does this correspond? (Substitute in the formula.)

6. The following substances melt at the temperatures indicated. To what Fahrenheit temperatures do these correspond?

Paraffin	+ 55° C.	Iron	+ 1200° C.
Tin	+ 232° C.	Mercury	− 39° C.

7. Attempts have been made to get record-breaking low temperatures. The following table gives low temperatures produced, the name of the experimenter, and the date of the experiment. To what Fahrenheit temperatures do these correspond?

DATE	TEMPERATURE	EXPERIMENTER
1714	− 17° C.	Fahrenheit
1823	− 102° C.	Faraday
1898	− 262° C.	Dewar
1908	− 269° C.	Onnes

8. The temperatures at three places in the United States on a certain day were:

(*a*) + 50° F.; (*b*) + 12° F; (*c*) − 8° F

What would these temperatures be on a Centigrade scale?

9. The following liquids boil at the temperatures indicated:

Alcohol 172.4° F. Turpentine 320° F.

Give the boiling temperatures on the Centigrade scale.

10. Air can be liquefied by reducing its temperature until it reaches − 182° C. To what Fahrenheit temperature does this correspond?

VIII. SPECIAL PRODUCTS AND FACTORING

85. In arithmetic, it is found necessary to memorize the multiplication table as an aid in multiplication, division, and factoring. In algebra, also, certain forms of number expressions occur frequently, which must be multiplied, divided, or factored by inspection.

86. **To Factor** an algebraic expression is to find two or more expressions which will produce the given expression when they are multiplied together.

Review the definitions of factor (§ 28) and common factor (§ 11).

87. A number which has no factors except itself and unity is called a **Prime Number**; as, 3, a, and $x+y$.

A monomial is expressed in items of its *prime factors* thus:

$$12\,a^3b^2c = 2\cdot 2\cdot 3\cdot a\cdot a\cdot a\cdot b\cdot b\cdot c.$$

88. Squaring a Monomial.

Development. 1. What does x^2 mean? $(xy)^2$? $(2\,r^3s)^2$?

2. Find each of the following squares by multiplication:

(*a*) $(2\,xy)^2$; (*b*) $(3\,a^2b^2)^2$; (*c*) $(-2\,r^3s^4)^2$.

3. Compare the exponent of each letter of the square with the exponent of that letter in the given monomial.

Rule. — To square a monomial:

Square its numerical coefficient, and multiply the result by each of the literal factors of the monomial, giving each letter twice its original exponent.

89. Cubing a Monomial.

Development. 1. Find each of the following cubes by multiplication:

(a) $(2x^2y)^3$; (b) $(3r^2s^3)^3$; (c) $(-2x^2y^2)^3$.

2. Compare the exponent of each letter of the cube with the exponent of that letter in the given monomial.

Rule. — To cube a monomial:

Cube its numerical coefficient, making the result negative if the given monomial is negative, and multiply the result by the literal factors of the monomial, giving each letter three times its original exponent.

Example 1. Find $(-5x^2y^3)^2$.

Solution: $(-5x^2y^3)^2 = +25x^4y^6$.

Example 2. Find $(-5x^2y^3)^3$.

Solution: $(-5x^2y^3)^3 = -125x^6y^9$.

EXERCISE 51

1. What sign does the square of any number have?
2. What sign does the cube of a negative number have?
3. Learn thoroughly the squares of the numbers from 1 to 20.
4. Learn thoroughly the cubes of the numbers from 1 to 6.

Give the values of the following indicated powers:

5. $(a^2b)^2$.
6. $(-a^3b^2)^2$.
7. $(2x^3y^2)^2$.
8. $(+abc)^3$.
9 $(+2a^2b)^3$.
10. $(+3x^3y^2z)^3$.
11. $(-5m^5)^2$.
12. $(-2x^2y)^3$.
13. $(-3x^4y^2)^3$.
14. $(-6x^2y^3)^2$.
15. $(-7x^3y^5)^2$.
16. $(+8a^3b^4)^2$.
17. $(-5m^2n^4)^3$.
18. $(-9r^2st^3)^2$.
19. $(+4c^5d^2)^3$.
20. $(11ab^2c^3)^2$.
21. $(-12m^3n^2)^2$.
22. $(+5m^4n^3)^3$.
23. $(-15v^4z^3)^2$.
24. $(-10c^4d^6)^3$.
25. $(-18)^2$.
26. $(-16t^3)^2$.
27. $(+6mn)^3$.
28. $(\frac{1}{2}a)^2$.
29. $(\frac{1}{3}mn)^3$.
30. $(-\frac{1}{2}nx^2)^3$.
31. $(+\frac{3}{5}a^2b)^2$.
32. $(-\frac{2}{3}cd^2)^3$.
33. $(-\frac{4}{5}xy)^3$.
34. $(-\frac{5}{8}r^2s)^2$.

90. The Square Root of a Monomial. If an expression can be resolved into two equal factors, it is said to be a **Perfect Square**, and one of the factors is said to be its **Square Root**.

Thus, $4a^2b^6$ is equal to $2ab^3 \times 2ab^3$; hence it is a perfect square and $2ab^3$ is its square root.

NOTE. $4a^2b^6$ is also equal to $(-2ab^3) \times (-2ab^3)$, so that $-2ab^3$ is also a square root. In this chapter, only the positive square root will be considered.

The following questions lead to the rule for extracting the square root of a perfect square monomial.

DEVELOPMENT. 1. What sign does the square of any monomial have?

2. When squaring a monomial, what do you do with the exponents of the literal factors? with the coefficient?

3. In finding the square root, then, what should you do with the exponents of the literal factors? with the coefficient?

4. Find the square root of each of the following monomials, and test the result by multiplication:

(*a*) x^4; (*b*) $4x^2y^2$; (*c*) $16r^2s^4$; (*d*) $25x^2y^2z^6$.

Rule.—1. A perfect square monomial is positive, has a perfect square numerical coefficient, and only even numbers as exponents.

2. To find its square root: find the square root of its numerical coefficient, and multiply the result by the literal factors of the monomial, giving each letter one half of its original exponent.

The symbol for extracting the square root is the **Radical Sign**, $\surd$; the vinculum is usually combined with it, $\sqrt{\ }$.

EXAMPLE. Find the square root of $25m^4n^6$.

SOLUTION: $\sqrt{25m^4n^6} = 5m^2n^3$.

91. The Cube Root of a Monomial. If an expression can be resolved into three equal factors, it is said to be a **Perfect Cube**, and one of the factors is said to be its **Cube Root**.

Thus, since $27a^6b^3$ is equal to $3a^2b \cdot 3a^2b \cdot 3a^2b$, it is a perfect cube, and $3a^2b$ is its cube root.

The following questions lead to the rule for extracting the cube root of a perfect cube monomial.

Development. 1. What sign does the cube of a positive number have? of a negative number?

2. When cubing a monomial, what do you do with the exponents of the literal factors? with the coefficient?

3. In finding the cube root, then, what should you do with the exponents of the literal factors of the monomial? with the coefficient?

4. Find the cube root of each of the following monomials and test the result by multiplication:

(a) a^3; (b) x^6; (c) y^9; (d) m^3n^3;
(e) $8a^3$; (f) $27m^3$; (g) $-c^6$; (h) $-64x^3y^3$.

Rule. — 1. A perfect cube monomial has a perfect cube numerical coefficient, whose sign may be + or −, and its literal factors have exponents which are exactly divisible by 3.

2. To find its cube root: find the cube root of the numerical coefficient, making it positive or negative, according as the sign of the monomial is + or −; and multiply the result by the literal factors of the monomial, giving each letter one third of its original exponent.

The symbol for extracting the cube root is the radical sign with the **Index** 3, as follows: $\sqrt[3]{\ }$.

Example. Find the cube root of $-125a^6b^9$.

Solution: $\sqrt[3]{-125a^6b^9} = -5a^2b^3$.

EXERCISE 52

Find the indicated roots:

1. $\sqrt{4m^2}$.
2. $\sqrt{9m^6}$.
3. $\sqrt{25a^2b^2}$.
4. $\sqrt[3]{a^6b^9}$
5. $\sqrt[3]{8a^3b^6}$.
6. $\sqrt[3]{27m^3n^3}$.
7. $\sqrt{36a^2b^6}$
8. $\sqrt[3]{-8a^3b^3}$.
9. $\sqrt[3]{-125m^6n^{12}}$.
10. $\sqrt{169m^6n^8}$.
11. $\sqrt[3]{-64x^3y^9}$.
12. $\sqrt{225x^{10}y^2}$.
13. $\sqrt{256r^2t^4}$.
14. $\sqrt[3]{+216c^3d^6e^3}$.
15. $\sqrt[3]{-27r^3t^{15}}$.

16. $\sqrt[3]{+125\,R^3}$.

17. $\sqrt{\frac{1}{4}a^2}$.

18. $\sqrt{\frac{1}{9}m^2n^2}$.

19. $\sqrt[3]{-216\,v^3t^3}$.

20. $\sqrt{\frac{25}{9}c^4d^2}$.

21. $\sqrt{\dfrac{1}{49\,x^2}}$.

22. $\sqrt{\dfrac{4\,a^2}{25\,c^2}}$.

23. $\sqrt[3]{\frac{1}{27}}$.

24. $\sqrt{\frac{1}{4}a^2b^2}$.

25. $\sqrt{\frac{9}{25}m^2n^4}$.

26. $\sqrt{\dfrac{m^2}{16}}$.

27. $\sqrt[3]{\frac{1}{8}a^3}$.

28. $\sqrt[3]{\frac{1}{27}m^3n^3}$.

29. $\sqrt[3]{\frac{64}{125}x^6}$.

30. $\sqrt{\dfrac{25\,a^4b^2}{49\,c^2}}$.

31. $\sqrt{\dfrac{256\,m^4}{169\,n^8}}$.

92. It is not always possible to factor a polynomial. Those polynomials which can be factored are the products of certain special forms of number expressions.

CASE I

Type Form: $a(b+c)=ab+ac$.

93. Multiplication. The rule for multiplying a polynomial by a monomial is given in § 57.

EXERCISE 53

Find:

1. $2\,x^2(x^2-3\,xy+y^2)$.
2. $-3\,xy(x^3+xy-y^3)$.
3. $+3\,mn(m^3-m^2n+n^3)$.
4. $+5\,a(3\,x^2-2\,xy+y^2)$.
5. $ab(3\,a^2-2\,ab+b^2)$.
6. $-3\,a(2\,a-b+c)$.

94. Factoring a Polynomial whose Terms have a Common Monomial Factor.

EXAMPLE 1. Factor $ax+3\,a-ba$.

SOLUTION: 1. Each term has the factor a. Divide the expression by a.

2. Then $\qquad ax+3\,a-ba=a(x+3-b)$.

CHECK: $\qquad a\cdot(x+3-b)=ax+3\,a-ab$.

EXAMPLE 2. Factor $14\,xy^4-35\,x^3y^2$.

SOLUTION: 1. Each term contains the factor $7\,xy^2$.

2. Dividing the expression by $7\,xy^2$, the quotient is $2\,y^2-5\,x^2$.

x $x^2 + y$
$x^3 + 4x$

3. Whence $14\,xy^4 - 35\,x^3y^2 = 7\,xy^2(2\,y^2 - 5\,x^2)$.

Check: $7\,xy^2(2\,y^2 - 5\,x^2) = 14\,xy^4 - 35\,x^3y^2$.

Rule. — **To factor a polynomial whose terms have a common monomial factor:**

1. **Find the greatest common factor of its terms.**
2. **Divide the polynomial by it.**
3. **The factors are the common factor found in step 1, and the quotient obtained in step 2.**

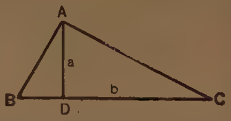
EXERCISE 54

Factor the following polynomials:

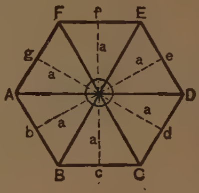
1. $3\,x + 3\,y$.
2. $4\,m - 4\,a$.
3. $2\,r - 6\,s$.
4. $5\,xt - 10\,xs$.
5. $3\,ax^2 - 2\,ay^2$.
6. $2\,rm^3 - 16\,rn^3$.
7. $x^3 + 4\,x$.
8. $3\,m^2 - 6\,m$.
9. $x^4 - x^3$.
10. $30\,r^2 - 5\,r^3$. $r(30 - 5r^2$
11. $2\,m^2 + 4\,mn + 2\,n^2$. $2(m^2 + 2mn +$
12. $x^2y + xy^2 + y^3$. $y(x^2 +$
13. $4\,a^2 - 8\,ab + 4\,b^2$. $4(a^2 -$
14. $3\,x^2 - 3\,xy + 3\,y^2$. $3(x^2 - xy +$
15. $r^2x^2 - 2\,r^2xy + r^2y^2$. $r(rx^2 - 2rx$
16. $12\,a^4 - 20\,a^3 + 4\,a^2$. $4(3a^4 - 5a^3 +$
17. $3\,x^2 - 15\,x + 18$.
18. $ar^6 - 5\,ar^3 + 6\,a$
19. $49\,m^4t^5 - 16\,n^2t^5$. $t^5(49m^4 - 16$
20. $15\,b^2 - 6\,b^5$.
21. $4\,a^2b^3 - 20\,ab^3x + 25\,b^3x^2$.
22. $-2\,a^2b + 16\,ab - 32\,b$. $b(-2a^2$
23. $3\,n^2y^3 - 21\,ny^3 + 18\,y^3$. $(n^2y^3 - 7$
24. $5\,ax^2y - 5\,axy - 30\,ay$.
25. $4\,abx^2 + 16\,abxy - 20\,aby^2$.
26. $9\,m^2t - 6\,mt - 63\,t$. $3t(m^2 - 3m -$
27. $48\,x^4y^2 - 144\,x^3y^3 + 108\,x^2y^4$. $12x(4x^3y - 12x^2y^2 +$
28. $x^5 - x^4 + x^3 - x^2$. $x(x^4 - x^3 + x^2 - x)$
29. $3\,am^2 - 6\,amn + 3\,an^2 - 3\,ap^2$. $3a(m^2 - 2mn + n^2 -$
30. $5\,ax^3 - 15\,ax^2y + 15\,axy^2 - 5\,ay^3$. $5a(x^3 - 3x^2y + 3$
$- y^3$

31. Recall that the area of a triangle is $\frac{1}{2}$ of the product of its base and altitude. Thus, the area of $\triangle ABC = \frac{1}{2} a \cdot b$. (§ 17)

Indicate the area of a triangle of base m and altitude p.



32. Suppose that the polygon $ABCDEF$ can be divided into six triangles, such that their altitudes are all equal. Call the altitudes each a, and the bases b, c, d, e, f, and g.



(*a*) What is the area of $\triangle OBC$? $\triangle ODC$? $\triangle ODE$? etc.?

(*b*) Indicate the sum of these areas.

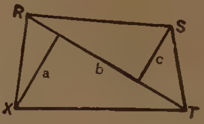
(*c*) Simplify that sum by removing the monomial factor.

(*d*) Simplify the result by substituting p for

$$(b+c+d+e+f+g).$$

The final result should be: area $= \frac{1}{2} ap$.

33. Suppose that the altitude of $\triangle RXT$ is a, and the altitude of $\triangle RST$ is c. The base of each is b.



(*a*) Represent the area of each.

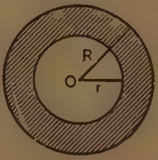
(*b*) Indicate the sum of these areas.

(*c*) Simplify the result by removing the monomial factor.

34. The area of a circle whose radius is r is πr^2.

(*a*) What is the area of the circle of radius R?

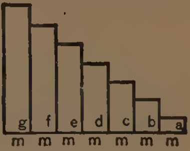
(*b*) How can you find the area of the ring between the large and small circles? Indicate this area.



(*c*) Simplify the result of step *b* by removing the monomial factor.

(*d*) Find the value of the result when R is 5 and r is 4.

35. Suppose that, in the adjoining figure, the rectangles have equal bases of length m, and altitudes of length a, b, c, etc.



(*a*) Represent the area of each.

(*b*) Indicate the sum of these areas.

(*c*) Simplify the resulting expression by removing the monomial factor.

to here

CASE II

Type Form: $(a+b)^2 = a^2 + 2ab + b^2$.

95. The Square of a Binomial.

DEVELOPMENT. 1. What does $(a+5)^2$ mean?

2. Find the value of the following by actual multiplication, as in § 58, and write the results as in part (*a*):

(*a*) $(a+6)^2 = a^2 + 12a + 36$.

(*b*) $(b+4)^2 = ?$

(*c*) $(m+5)^2 = ?$

(*d*) $(x+8)^2 = ?$

3. Observe carefully the results in step 2. Then try to find the following squares mentally, first, and check by multiplication:

(*a*) $(x+2)^2 = ?$

(*b*) $(y+3)^2 = ?$

(*c*) $(z+7)^2 = ?$

(*d*) $(k+10)^2 = ?$

4. Write the sum of x and y. Indicate the square of that sum. Find the value of the square either mentally or by multiplication.

5. Prove by multiplication the following fact:

$$(a+b)^2 = a^2 + 2ab + b^2.$$

Rule. — **To square the sum of two numbers:**

Square the first number; add twice the product of the two numbers; add the square of the second number.

EXAMPLE. Square $(3a + 2bc)$.

SOLUTION: $$(3a + 2bc)^2 = (3a)^2 + 2(3a)(2bc) + (2bc)^2$$
$$= 9a^2 + 12abc + 4b^2c^2.$$

The solution may be checked by substitution, but it is necessary to acquire such skill in doing these problems that checking in that manner will be unnecessary.

Rule. — To square the difference of two numbers:

Square the first number; subtract twice the product of the two numbers; add the square of the second number.

EXAMPLE. Square $(4a^2 - 5b^3)$.

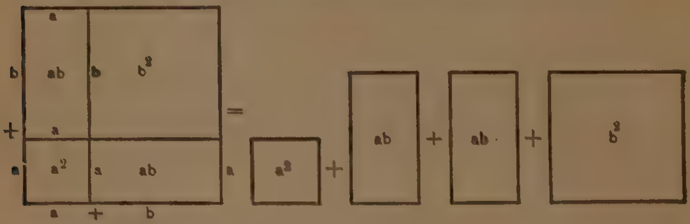
SOLUTION: $(4a^2 - 5b^3)^2 = (4a^2)^2 - 2(4a^2)(5b^3) + (5b^3)^2$
$= 16a^4 - 40a^2b^3 + 25b^6$.

NOTE. In actual practice, pupils should do all of this work mentally, passing from the given problem to the result as follows:

$$(3m - 5n)^2 = 9m^2 - 30mn + 25n^2.$$

This is called "finding the result by *inspection*."

The following figure illustrates the square of $(a + b)$.



$$(a + b)^2 = a^2 + 2ab + b^2.$$

EXERCISE 55

Square the following binomials by inspection:

1. $a + 5$.
2. $b + 6$.
3. $c - 7$.
4. $d - 3$.
5. $m^2 + 4$.
6. $n^2 - 8$.
7. $p^3 + 9$.
8. $x^3 - 10$.
9. $r^2 + 12$.
10. $mn - 11$.
11. $2a + b$.
12. $3a - c$.
13. $a + 2b$.
14. $r - 3s$.
15. $m + 4n$.
16. $2p - 3q$.
17. $x^3 - 3y$.
18. $2x + 5$.
19. $5m - 6$.
20. $3a^2 - 2b^2$.
21. $2xy + 9$.

22. $3a^3 - 6b$. **25.** $9a^2 + 5r^4$. **28.** $2xy - 9z^2$.
23. $10r + 4t^2$. **26.** $7 - 2a^2$. **29.** $7ab - 5cd$.
24. $11s - 5t$. **27.** $8c + 3m^2n^3$. **30.** $9a^4 + 6b^2$.

Note. For additional drill problems, if desired, square a binomial like $3x + b$, making b successively 1, 2, 3, etc. up to 10. Then change 3 to 4 or any other number. Short daily drills of this sort afford good mental arithmetic.

Expand the following:

31. $(m - \frac{3}{5})^2$. **34.** $(x + \frac{1}{3})^2$. **37.** $(r + \frac{1}{6})^2$.
32. $(y - \frac{1}{5})^2$. **35.** $(n + \frac{1}{2})^2$. **38.** $(s + \frac{2}{7}t)^2$.
33. $(z + \frac{2}{3})^2$. **36.** $(p - \frac{3}{4})^2$. **39.** $(x - \frac{3}{8}y)^2$.

40. Square 29 mentally.

Solution: $29^2 = (30 - 1)^2 = 900 - 60 + 1 = 901 - 60 = 841$.

This should be done mentally.

41. Square 32. (Hint: $32 = 30 + 2$).

Square mentally the following numbers:

42. 21. **46.** 42. **50.** 49 **54.** 52.
43. 22. **47.** 33. **51.** 28. **55.** 43.
44. 23. **48.** 19. **52.** 39. **56.** 57.
45. 31. **49.** 18. **53.** 38.

57. *Problem.* Find a rule for squaring any number ending in 5.

Solution: 1. $35 = 3 \times 10 + 5$; $45 = 4 \times 10 + 5$; $55 = 5 \times 10 + 5$.

2. Similarly, any number ending in 5 may be represented by

$$10n + 5.$$

Thus, for 95, n is 9, since $9 \times 10 + 5 = 95$.

3.
$$(10n + 5)^2 = 100n^2 + 100n + 25$$
$$= 100n(n + 1) + 25,$$

or,
$$n \cdot (n + 1) \text{ hundreds} + 25.$$

Thus the square of 95, in which $n = 9$, is

$$9 \cdot (9 + 1) \text{ hundreds} + 25, \text{ or } 9025.$$

Rule.—**To square a number ending in 5, drop the 5, multiply the balance of the number by the consecutive integer, and affix 25 to the result.**

EXAMPLE $85^2 = 7225$.

I.e. $8 \times 9 = 72$; affixing 25, the result is 7225

58. Find by this rule the squares of some numbers ending in 5, such as 35, 45, 105, 115, etc.

96. Factoring Perfect Square Trinomials. In algebra, it is necessary to be able to recognize a perfect square trinomial.

DEVELOPMENT. 1. Square the following binomials, and write the result as in part (*a*):

$9x^2 -$

(*a*) $(a + b)^2 = a^2 + 2\,ab + b^2$.
(*c*) $(3\,x + 4\,y)^2 = ?$
(*b*) $(2\,a + 3\,b)^2 = ?$
(*d*) $(4\,m - 5\,n)^2 = ?$

2. How many terms are there always in the square?

3. What sign does the first term of the square have? the third term?

4 Notice that the first and third terms are perfect squares and that the second term is twice the product of the square roots of these two terms.

5. Are the following perfect squares? Give the reason for your opinion. Give the square roots of the perfect squares:

(*a*) $c^2 + 2\,cd + d^2$.
(*c*) $r^2 - 6\,r + 9$.
(*b*) $m^2 + 2\,mn + n^2$.
(*d*) $x^2 - 10\,x - 25$.

Rule.—**1. A trinomial is a perfect square when two of its terms are perfect squares and positive, and when the remaining term is twice the product of the square roots of the perfect square terms.**

2. To find the square root of a perfect square trinomial: extract the square roots of the two perfect square terms, and connect them by the sign of the remaining term.

EXAMPLE 1. Is $4\,x^2 + 9\,y^4 - 12\,xy^2$ a perfect square?

SOLUTION: $4\,x^2$ is a perfect square; its square root is $2\,x$.
$9\,y^4$ is a perfect square; its square root is $3\,y^2$.

$$12\,xy^2 = 2(2\,x)(3\,y^2).$$

$2x - 3x^2$

Hence $4x^2 + 9y^4 - 12xy^2$ is a perfect square, and its square root is $2x - 3y^2$.

EXAMPLE 2. Is $9a^2 + 7ab + 4b^2$ a perfect square?

SOLUTION. $9a^2$ and $4b^2$ are perfect squares. Their square roots are $3a$ and $2b$, respectively.

$2(3a)(2b) = 12ab$. Since the third term is $7ab$, and not $12ab$, $9a^2 + 7ab + 4b^2$ is not a perfect square.

To be a perfect square, the term $7ab$ would need to be changed to + or − $12ab$.

Then $$\sqrt{9a^2 \pm 12ab + 4b^2} = 3a \pm 2b.$$

EXERCISE 56

Supply the missing term so as to make perfect square trinomials of the following expressions, and then give the square roots:

1. $x^2 + (?) + m^2$.
2. $a^2 - (?) + t^2$.
3. $m^2 - (?) + n^2$.
4. $r^6 - (?) + t^2$
5. $x^4 - (?) + 9y^2$.
6. $9x^2 + (?) + 4y^2$.
7. $16r^4 - (?) + 25t^8$.
8. $100m^2 + (?) + 4n^6$.
9. $25c^2 - (?) + 9d^2$.
10. $81c^8 - (?) + 25d^2$.
11. $m^2 + 4m + (?)$.
12. $x^2 - 6x + (?)$.
13. $y^2 - 12y + (?)$.
14. $z^4 - 10z^2 + (?)$.
15. $9a^2 + 6a + (?)$.
16. $144A^2 - (?) + 25$.
17. $9b^2 - (?) + 36c^2$.
18. $25x^4 + (?) + 36y^2$.
19. $49r^2 - (?) + 25s^2$.
20. $9x^2 - (?) + 64$.

In the following, determine whether the trinomials are perfect squares; find the factors *when possible.*

21. $4m^2 - 20mn^2 + 25n^4$.

SOLUTION: 1. This is a perfect square, according to the rule (§ 96, Rule 1.)

2. Hence, $4m^2 - 20mn^2 + 25n^4 = (2m - 5n^2)^2$
$= (2m - 5n^2)(2m - 5n^2)$.

22. $m^2 - 10\,mn + 25\,n^2$.

23. $x^4 + 12\,x^2y + 36\,y^2$.

24. $m^2x^2 + 18\,mx + 81$.

25. $64\,a^2 + 15\,ab + b^2$.

26. $100\,x^4 - 60\,x^2y + 9\,y^2$.

27. $49\,x^2y^2 - 70\,xyz + 25\,z^2$.

28. $4\,a^2 - 22\,ax + 25\,x^2$.

29. $81\,x^2 + 16\,y^2 - 72\,xy$.

30. $4\,a^2 - 28\,ax + 49\,x^2$.

31. $25\,x^2 + 16\,y^2 - 40\,xy$.

32. $9\,m^2n^2 + 25\,r^4 - 30\,mnr^2$

33. $4\,t^4 + 36\,x^2 - 12\,xt^2$.

34. $169\,m^6 - 26\,m^3n + n^2$.

35. $64\,a^2b^2 + 16\,abcd + c^2d^2$.

36. $100\,x^8 - 80\,x^4 + 16$.

37. $49\,m^4 + 112\,m^2 + 64$.

38. $9\,a^2 + 42\,ab + 49\,b^2$.

39. $121\,a^2b^2 + 130\,abc + 36\,c^2$.

40. $64\,a^2 + 176\,ab + 121\,b^2$.

41. $x^2 + x + \frac{1}{4}$.

42. $y^2 + \frac{2}{3}\,y + \frac{1}{9}$.

43. $z^2 + \frac{4}{25}\,w^2 - \frac{4}{5}\,zw$.

44. $\frac{1}{9}\,m^2 + 25\,n^2 - \frac{10}{3}\,mn$.

45. $\frac{4}{25}\,x^2 + \frac{12}{5}\,xy + 9\,y^2$.

97. Complete Factoring. When a number is to be factored, all of its prime factors should be found. The factors found first may sometimes be factored again.

Thus, $48 = 8 \cdot 6 = 4 \cdot 2 \cdot 2 \cdot 3 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3$.

In algebra, this sort of factoring is frequently necessary.

EXAMPLE 1. Find the prime factors of

$$5\,am^2 - 50\,amn + 125\,an^2.$$

SOLUTION:

$$\begin{aligned} &5\,am^2 - 50\,amn + 125\,an^2 \\ &= 5\,a(m^2 - 10\,mn + 25\,n^2) && (\S\ 94) \\ &= 5\,a(m - 5\,n)(m - 5\,n). && (\S\ 96) \end{aligned}$$

Do not fail to rewrite all factors, like the $5\,a$, which are not factored again.

Rule. — To find the prime factors of an expression:

1. First remove any monomial factor which may be present.

2. Then factor the resulting expression, when possible, rewriting all expressions which cannot be factored.

EXERCISE 57

Find all of the prime factors:

1. $3mx^2 + 24mx + 48m$.
2. $18t - 12at + 2a^2t$.
3. $7m^2n^3 + 70mn^4 + 175n^5$.
4. $5x^2y^2z + 70xyz + 245z$.
5. $11mx^2y^2 + 22mxy + 44m$.
6. $20a^6 - 20a^3bc^2 + 5b^2c^4$.
7. $30a^2b - 120ab + 120b$.
8. $3ax^2 - 3axy + 3ay^2$
9. $75cx^2 + 48cy^2 - 120cxy$.
10. $5r^2s - 10rst + 20st^2$.

CASE III

Type Form: $(a+b)(a-b) = a^2 - b^2$.

98. The Product of the Sum and the Difference of Two Numbers.

DEVELOPMENT. 1. Find by multiplication the following products, and write the results as in part (*a*):

(*a*) $(x+3)(x-3) = x^2 - 9$.
(*b*) $(m+7)(m-7) = ?$
(*c*) $(k+10)(k-10) = ?$
(*d*) $(r+9)(r-9) = ?$

2. Observe the results in step 1; try to find the following products mentally. Check the results by multiplication.

(*a*) $(a+6)(a-6) = ?$
(*b*) $(c+8)(c-8) = ?$
(*c*) $(d+4)(d-4) = ?$
(*d*) $(y+5)(y-5) = ?$

3. Write the sum of x and y; write their difference. Find the product of the results.

4. Prove by multiplication the following fact:

$$(a+b)(a-b) = a^2 - b^2$$

Rule. — To find the product of the sum and the difference of two numbers:

1. Square each of the numbers.

2. Subtract the second square from the first.

EXAMPLE 1. Find $(5a^2 + m)(5a^2 - m)$.

SOLUTION: $(5a^2 + m)(5a^2 - m) = (5a^2)^2 - (m)^2 = 25a^4 - m^2$.

EXAMPLE 2. Find mentally the product of 24 and 16.

SOLUTION. $24 \times 16 = (20 + 4)(20 - 4) = 400 - 16 = 384$.

$\frac{1}{2} \times \frac{2}{3} = \frac{1}{9}$

EXERCISE 58

Find by inspection the products:

1. $(a+2)(a-2)$.
2. $(r-3)(r+3)$.
3. $(s^2+4)(s^2-4)$.
4. $(x^2+5y)(x^2-5y)$.
5. $(3m+4n)(3m-4n)$.
6. $(m^4-1)(m^4+1)$.
7. $(z-3a)(z+3a)$.
8. $(8k^2-9t)(8k^2+9t)$.
9. $(3ab-7c)(3ab+7c)$.
10. $(4r-5s^2)(4r+5s^2)$.
11. $(10xy-11)(10xy+11)$.
12. $(13m^2-12)(13m^2+12)$.
13. $(a^5-b^5)(a^5+b^5)$.
14. $(x^3-y^3)(x^3+y^3)$.
15. $(x^6-8y^6)(x^6+8y^6)$.
16. $(7xy^2z-10)(7xy^2z+10)$.
17. $(\frac{2}{3}x-\frac{1}{5})(\frac{2}{3}x+\frac{1}{5})$.
18. $(\frac{3}{4}m+\frac{1}{7})(\frac{3}{4}m-\frac{1}{7})$.
19. $(\frac{2}{9}r^2-\frac{3}{11})(\frac{2}{9}r^2+\frac{3}{11})$.
20. $(\frac{7}{8}n^2-\frac{5}{4})(\frac{7}{8}n^2+\frac{5}{4})$.

21. $(9+5)(9-5)$.
22. $(25+2)(25-2)$.
23. $22 \cdot 18$.
24. $23 \cdot 17$.
25. $32 \cdot 28$.
26. $53 \cdot 47$.
27. $62 \cdot 58$.
28. $98 \cdot 102$.
29. $55 \cdot 65$.
30. $33 \cdot 37$.
31. $41 \cdot 49$.
32. $22 \cdot 28$.

33. Find the cost of 18 dozen of eggs at 22¢ per dozen.
34. Find the cost of 16 yards of gingham at 24¢ per yard
35. Find the cost of 45 yards of scrim at 55¢ per yard.

99. Factoring the Difference of Two Perfect Squares.

DEVELOPMENT. 1. What is the product of $(a+2)$ and $(a-2)$? What, then, are the factors of a^2-4?

2. Find the factors of:

(a) a^2-9;
(b) m^2-16;
(c) k^2-l^2;
(d) $9r^2-4s^2$.

3 Write the square of r; of t; the difference of these squares. Factor the result.

4. Similarly, the factors a^2-b^2 are:

$$(a^2-b^2)=(a+b)(a-b).$$

Rule. — To factor the difference of two squares:

1. Find the square root of the two perfect square terms.

2. One factor is the sum of the results; the other factor is the difference of the results.

EXAMPLE 1. Find the factors of $25\,r^4 - 16\,t^6$.

SOLUTION: $25\,r^4 - 16\,t^6 = (5\,r^2)^2 - (4\,t^3)^2$
$= (5\,r^2 + 4\,t^3)(5\,r^2 - 4\,t^3)$.

EXAMPLE 2. Find mentally the value of $13^2 - 7^2$.

SOLUTION: $13^2 - 7^2 = (13 + 7)(13 - 7)$
$= 20 \times 6 = 120$.

This example shows how the above rule can be used to simplify arithmetical work.

EXERCISE 59

Factor the following *when possible:*

1. $a^4 - b^2$.
2. $c^2 - 9$.
3. $16 - d^2$.
4. $x^2 - 1$.
5. $1 - y^4$
6. $x^2 - 4\,y^2$.
7. $9 - m^2$.
8. $r^4s^2 - t^2$.
9. $16\,x^2 - 25\,y^2$.
10. $81\,r^2 - 16\,z^4$.
11. $25\,a^6 - 81\,z^8$.
12. $36 - 49\,y^4$.
13. $1 - 36\,a^2b^2$.
14. $x^2 - \frac{1}{4}$.
15. $100\,z^4x^2 - 49$.
16. $y^6 - \frac{1}{9}\,m^2$
17. $\frac{4}{9}\,a^2 - \frac{1}{36}\,c^2$.
18. $64\,m^4 - 81\,n^6$.
19. $169\,a^2 - 196$.
20. $25\,a^3 - 1$.
21. $144\,x^8 - 121\,y^2$.
22. $\frac{4}{25}\,b^2 - \frac{1}{49}$.
23. $100\,a^2b^2c^2 - 1$.
24. $c^4 - \frac{4}{9}\,d^2$.
25. $81\,x^{10} - 196\,y^6z^2$.
26. $256\,n^4 - m^2$.
27. $225 - 64\,t^2$.
28. $85\,x^2 - y^2$.
29. $\dfrac{1}{a^2} - \dfrac{c^2}{d^2}$.
30. $\dfrac{25}{r^2} - \dfrac{s^2}{36}$.

Find mentally the following:

31. $16^2 - 9^2$.
32. $23^2 - 7^2$.
33. $24^2 - 6^2$.
34. $33^2 - 17^2$.
35. $24^2 - 16^2$.
36. $35^2 - 15^2$.
37. $27^2 - 13^2$.
38. $26^2 - 4^2$.
39. $95^2 - 5^2$.
40. $75^2 - 25^2$.

Find the prime factors of the following:

41. $5x^2 - 5y^2$. (See § 94.)

42. $m^2t - 25t$.

43. $3ar^2 - 12as^4$.

44. $4c^2d - 9d^3$.

45. $32y - 2x^4y$.

46. $36xw^2 - xv^2$.

47. $\pi R^2 - \pi s^2$.

48 $\frac{1}{4}\pi d^2 - \frac{1}{4}\pi m^2$.

49. $m^4 - n^4$.

50. $x^8 - y^8$.

Find the following quotients:

51. $(x^2 - y^2) \div (x - y)$.

52. $(a^2 - m^2) \div (a + m)$.

53. $(r^2 - 9) \div (r - 3)$.

54. $(l^2 - 25) \div (l + 5)$.

55. $(4s^2 - t^2) \div (2s - t)$.

56. $(9x^2 - 16y^2) \div (3x + 4y)$.

57. $(25m^2 - 16n^2) \div (5m + 4n)$.

58. $(169 - 100a^2) \div (13 - 10a)$.

Tell what binomial will divide each of the following; give the quotient:

59 $9x^2 - 4y^4$.

60. $m^2 - 16n^2$.

61. $25a^6 - 36b^4$.

62. $144r^6 - 121$.

63. $256c^4 - 400$.

64. $100r^2 - 36t^4$.

EXERCISE 60

Review

Expand the following:

1. $(x^2 - y^2)^2$.

2. $2a(x + y)(x - y)$.

3. $3m(x^3 - y^3)(x^3 + y^3)$.

4. $2pg(r^2 - s^2)(r^2 + s^2)$.

5. $3c(2a - b)^2$.

6. $5ab(a^2 - b^2)^2$.

Factor completely the following:

7. $r^4 - 2r^2s^2 + s^4$.

8. $mc^4 - 2mc^2 + m$.

9. $5tx^4 - 5ty^4$.

10. $3ax^4 - 6ax^2y^2 + 3ay^4$.

11. $5mx^2 - 10mxy + 5my^2$

12. $2r^3m^4 - 4r^3m^2n^2 + 2r^3n^4$.

CASE IV

Type Form: $(x+a)(x+b)=x^2+(a+b)x+ab.$

100. The Product of Two Binomials having a Common Term.

DEVELOPMENT. 1. Find by actual multiplication the following products, and write the results as in part (*a*):

(*a*) $(x+2)(x+3)=x^2+5x+6.$

(*b*) $(x+5)(x+3)=?$ (*d*) $(m-7)(m-2)=?$

(*c*) $(a+5)(a+6)=?$ (*e*) $(s-5)(s-8)=?$

2. Observe carefully the results in 1. Try to find the following products mentally; check the results by multiplication:

(*a*) $(b+4)(b+2).$ (*c*) $(x-3)(x-7).$

(*b*) $(c+4)(c+3).$ (*d*) $(y-4)(y-5).$

Rule.—To obtain the product of two binomials having a common term:

1. Square the common term.

2. Multiply the common term by the algebraic sum of the second terms of the binomials.

3. Find the product of the second terms.

4. Add the results.

EXAMPLE 1. Find the product of $x-8$ and $x+5$.

SOLUTION: $(x-8)(x+5)=x^2+(-8+5)x+(-8)(+5)$

$=x^2-3x-40.$

EXAMPLE 2. Find $(ab+2)(ab-11)$.

SOLUTION: $(ab+2)(ab-11)=a^2b^2-9ab-22.$

NOTE. Here, one glances at $+2$ and -11, notes that their sum is -9 and that their product is -22, and writes the result as above.

EXAMPLE 3. Find mentally the product of 23 and 24.

SOLUTION: 1. $23\times 24=(20+3)(20+4)=400+7\cdot 20+12$

$=400+140+12=552$

EXERCISE 61

Find the following products:

1. $(x+2)(x+3)$.
2. $(x+2)(x+4)$.
3. $(a+3)(a+5)$.
4. $(a+4)(a+6)$.
5. $(m+5)(m+9)$.
6. $(b-3)(b-7)$.
7. $(b-4)(b-8)$.
8. $(c-5)(c-7)$.
9. $(r-6)(r-8)$.
10. $(r-9)(r-10)$.
11. $(r+2)(r-8)$.
12. $(m+3)(m-10)$.
13. $(n+5)(n-11)$.
14. $(s+6)(s-12)$.
15. $(t+4)(t-11)$.
16. $(x-2)(x+10)$.
17. $(y+11)(y-3)$.
18. $(z+10)(z-5)$.
19. $(w-4)(w+9)$.
20. $(a-1)(a+8)$.
21. $(s^2-6)(s^2+10)$.
22. $(t^3-12)(t^3+9)$.
23. $(xy-10)(xy+15)$.
24. $(r-5s)(r+3s)$.
25. $(x-2y)(x+8y)$.
26. $(rs-8t)(rs-3t)$.
27. $(a+3b)(a+13b)$.
28. $(c-4d)(c-12d)$.
29. $(x^2-8y)(x^2-10y)$.
30. $(x^3+5y^3)(x^3-2y^3)$.
31. $(p^2+11q)(p^2+15q)$.
32. $(r^3-12)(r^3+20)$.
33. $(s^3-15)(s^3+8)$.
34. $(t^2-16w)(t^2-4w)$.
35. $(x^3-18y)(x^3+5y)$.
36. $(a+17t^2)(a-3t^2)$.
37. $(x-9a)(x+15a)$.
38. $(x+11b)(x-8b)$.
39. $(y^2-13z)(y^2-7z)$.
40. $(t+15r)(t-10r)$.
41. $(a^2-14b)(a^2-10b)$.
42. $(c-22x)(c+20x)$.
43. $(x+19y)(x-5y)$.
44. $(a-33b)(a+6b)$.
45. $(c^2+25d)(c^2-10d)$.
46. $(x-35)(x+20)$.
47. $(m-\frac{1}{5})(m-5)$.
48. $(z+\frac{1}{2})(z+1)$.
49. $(y+\frac{1}{3})(y+3)$.
50. $(x+\frac{1}{2})(x+\frac{1}{4})$.

51. Find mentally the product of 62×68.

SOLUTION: $62 \times 68 = (60+2)(60+8) = 3600 + 600 + 16 = ?$

Find mentally the following arithmetical products:

52. 22×28.

53. 33×37.

54. 34×36.

55. 24×26

56. 32×38.

57. 44×46.

58. 52×54.

59. 23×25.

60. 33×34.

Perform mentally the multiplications in the following problems:

61. Find the area of a rectangle whose base is 26 inches and altitude is 24 inches.

62. Find the cost of 12 yards of lawn at 18 ¢ per yard.

63. Find the cost of 22 yards of scrim at 25 ¢ per yard.

64. Find the cost of 53 bushels of corn at 55 ¢ per bushel.

65. Find the cost of an 11-pound roast at 18 ¢ per pound.

101. Factoring Trinomials of the Form x^2+px+q

A trinomial of the form x^2+px+q can be factored if it is the product of two binomials having a common term (§ 100).

DEVELOPMENT. 1. $(x+5)(x-3)=x^2+2x-15$.

In obtaining this product, the algebraic sum of $+5$ and -3 is taken for the coefficient of x, and the product of $+5$ and -3 is taken for the third term.

To factor $x^2+2x-15$, it is necessary to find two numbers whose product is -15, and whose algebraic sum is $+2$.

2. Factor $x^2+7x+12$.

Two numbers whose product is $+12$, are $+3$ and $+4$, and their sum is $+7$. Try as the factors $(x+3)$ and $(x+4)$.

CHECK: Does $(x+3)(x+4)=x^2+7x+12$? Yes.

Rule. — To factor a trinomial of the form x^2+px+q

1. Find two numbers whose algebraic product is q and whose algebraic sum is p.

2. One factor is $x +$ one number; the other factor is $x +$ the other number.

Example 1. Factor $x^2-26x-192$.

Solution: It is necessary to find two numbers whose sum is -26 and whose product is -192; the number of greater absolute value must be negative. If necessary write all possible pairs of factors of -192, one of which is $+$ and the other $-$. We have:

$(+1)\times(-192)$; sum $=-191$.
$(+2)\times(-96)$; sum $=-94$.
$(+3)\times(-64)$; sum $=-61$.
$(+4)\times(-48)$; sum $=-44$.
$(+6)\times(-32)$; sum $=-26$.

$\therefore +6$ and -32 are the numbers required.

$$\therefore x^2-26x-192=(x+6)(x-32).$$

EXERCISE 62

Factor:

1. $x^2+14x+45$.

Solution: 1. Find two numbers whose product is 45 and whose sum is $+14$.

2. Factors of 45 are, 1 and 45; 3 and 15; 5 and 9.

3. $5+9=14$, hence the factors are $(x+5)(x+9)$.

Check: $(x+5)(x+9)=x^2+14x+45$.

Note. This solution should all be done mentally; decide upon a pair of factors of 45 and immediately determine their sum.

2. x^2+5x+6.
3. $x^2+8x+15$.
4. $r^2+10r+21$.
5. $a^2+11a+28$.
6. $m^2+9m+20$.
7. $t^2+9t+18$.

8. $p^2-12p+32$.

Solution: 1. Since 32 is positive, the second terms of the factors must have the same sign; since the sum of the second terms is -12, the second terms are negative.

2. $(-8)\times(-4)=+32$; and $(-8)+(-4)=-12$.

Therefore the factors are $(p-8)(p-4)$.

Check: $(p-8)(p-4)=p^2-12p+32$.

9. $x^2-7x+12$.
10. $y^2-9y+14$.
11. $z^2-11z+24$.
12. $w^2-10w+24$.
13. $a^2-11a+30$.
14. $c^2-12c+35$.

15. $x^2 + 6x - 16.$

SOLUTION: Since -16 is negative, the second terms of the factors must have unlike signs; since the sum of the second terms is $+6$, the term of greater absolute value must be positive.

2. Factors of -16 of this sort are: $(+16, -1)$ and $(+8, -2)$.

3. $(+8) + (-2) = +6$, therefore the factors are $(x + 8)$ and $(x - 2)$.

CHECK: $(x + 8)(x - 2) = x^2 + 6x - 16.$ (By § 100.)

16. $x^2 + 3x - 40.$

17. $x^2 + 2x - 24.$

18. $y^2 + 4y - 21.$

19. $z^2 + 4z - 32.$

20. $w^2 + 4w - 60.$

21. $a^2 + 3a - 54.$

22. $m^2 - 4m - 21.$

SOLUTION: 1. The factors of -21 must have unlike signs; the one of greater absolute value is negative, since the sum is -4.

2. Such factors of -21 are: -21 and $+1$; -7 and $+3$.

3. $(-7) + (+3) = -4$; therefore the factors are $(m - 7)$ and $(m + 3)$.

CHECK: $(m - 7)(m + 3) = m^2 - 4m - 21.$ (See § 100.)

23. $x^2 - 2x - 35.$

24. $n^2 - 7n - 18.$

25. $a^2 - 8a - 33.$

26. $r^2 - 4r - 45.$

27. $m^4 - 11m^2 + 30.$

28. $a^2 - 15a + 54.$

29. $r^2 + r - 2.$

30. $x^4 + 6x^2 + 8.$

31. $c^2 + 15cd + 36d^2.$

32. $x^2 - 12xy + 32y^2.$

33. $z^2 + 2z - 63.$

34. $c^2d^2 - cd - 6.$

35. $y^2 + 10yz + 9z^2.$

36. $x^2 - 13xw + 22w^2.$

37. $m^2 - 7m - 44.$

38. $r^4 + 2r^2s - 48s^2.$

39. $s^2 - 6st - 55t^2.$

40. $w^6 - 18w^3 + 72.$

41. $y^2 + 4y - 96.$

42. $a^2 - 70 - 3a.$ (Rearrange it.)

43. $b^2 - 24 - 10b.$

44. $c^2 + 20c + 84$

45. $z^2 - 36 + 5z.$

46. $a^4 + 25a^2 - 150.$

47. $-19m^2 + m^4 + 84.$

48. $x^4 + 50 - 27x^2.$

49. $t^2 - st - 20\,s^2$.

50. $a^2 + 7\,ab - 60\,b^2$.

51. $x^2 - 20\,ax + 99\,a^2$.

52. $r^2 + rs - 72\,s^2$.

53. $x^2 + 50\,y^4 - 15\,xy^2$.

54. $a^2 + 16\,a + 15$.

55. $a^2b^2 - 16\,abc + 28\,c^2$.

56. $x^2 - 21\,xyz + 108\,y^2z^2$.

57. $a^2 - 21\,ab^2 + 110\,b^4$.

58. $c^2d^2 - 10\,cd - 96$.

59. $a^2c^2 - 26\,ac + 160$.

60. $r^2 - 7\,rs - 120\,s^2$.

Find all of the prime factors:

61. $3\,ab^2 - 15\,ab + 18\,a$.

SOLUTION: $3\,ab^2 - 15\,ab + 18\,a = 3\,a(b^2 - 5\,b + 6)$

$= 3\,a(b-2)(b-3)$.

62. $5\,t^2 + 35\,t + 60$.

63. $mr^2 + 2\,mr - 15\,m$.

64. $3\,cx^2 + 6\,cx - 9\,c$.

65. $7\,y^2z + 21\,yz - 126\,z$.

66. $abn^2 - 7\,abn + 6\,ab$.

67. $11\,x^2 - 11\,x - 66$.

68. $8\,x^2 + 48\,x - 56$.

69. $m^3n^2 + 2\,m^3n - 63\,m^3$.

CASE V

Type Form $(ax + b)(cx + d)$.

102. The Product of Two Binomials of the Form $(ax + b)$.

DEVELOPMENT. 1. Find by actual multiplication the following products, and write the results as in part (*a*):

(*a*) $(2\,x + 3)(3\,x + 1) = 6\,x^2 + 11\,x + 3$.

(*b*) $(3\,m + 2)(2\,m + 3) = ?$ (*c*) $(2\,y + 4)(3\,y - 2) = ?$

2. Examine carefully the results in 1, then try to get the first and third terms of the following products *mentally*; possibly you can get the second term also. Check by actual multiplication.

(*a*) $(2\,r + 4)(3\,r + 1)$.

(*b*) $(2\,r + 5)(r + 2)$.

(*c*) $(3\,s + 5)(2\,s + 1)$.

(*d*) $(4\,x + 2\,y)(2\,x + 3\,y)$.

3. Below is given the product of $(5x-4y)$ and $(2x+3y)$.

$$\begin{array}{rrr}
5x & -4y & \\
2x & +3y & \\
\hline
10x^2 & -8xy & \\
 & +15xy & -12y^2 \\
\hline
10x^2 & +7xy & -12y^2
\end{array}$$

$$\therefore (5x-4y)\ (2x+3y)=10x^2+(-8+15)xy-12y^2.$$

Rule. — 1. The first term of the product is the product of the first terms of the binomials. $(5x \cdot 2x=10x^2.)$

2. The third term of the product is the product of the second terms of the binomials. $(-4y)\cdot(+3y)=-12y^2.$

3. The second term of the product is the algebraic sum of the "cross products." (Notice the position of these cross products on the left of the equation: $-4y \cdot 2x$ and $5x \cdot 3y$).

EXAMPLE 1. Find the product $(5r-6s)\ (2r+3s)$.

SOLUTION: In all of these examples, the only difficulty is that of finding the "middle term."

1. $5r \cdot 3s=15rs$, $-6s \cdot 2r=-12rs$, and $(15rs)+(-12rs)=+3rs.$
2. $\therefore (5r-6s)(2r+3s)=10r^2+3rs-18s^2.$

EXAMPLE 2. Find the product $(9x+4y)(3x-6y)$.

SOLUTION: $9\cdot-6=-54$, $4\cdot 3=12$, and $(-54)+(12)=-42.$

$$\therefore (9x+4y)(3x-6y)=27x^2-42xy-24y^2.$$

In this example, the coefficient of the middle term was found first and then xy was affixed. In practice all of this should be done mentally as in the following example.

EXAMPLE 3. Find the product $(7m-4n)(8m+5n)$.

SOLUTION: $(7m-4n)(8m+5n)=56m^2+3mn-20n^2.$

[Middle term: $7\cdot 5=35$, $-4\cdot 8=-32$, $35+(-32)=3$.]

Pupils should try to acquire such skill that they can find the correct products mentally. This is the manner in which experts do it.

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EXERCISE 63

1. $(2x+2)(x+3)$.
2. $(2x+3)(x+2)$.
3. $(2x+1)(x+3)$.
4. $(3x+1)(2x+3)$.
5. $(2x+3)(3x+2)$.
6. $(3a-2)(2a-1)$. 72
7. $(3a-4)(a-2)$.
8. $(3a-4)(a-3)$.
9. $(3a-5)(2a-3)$.
10. $(4a-5)(2a-5)$.
11. $(3r+5)(2r-1)$.
12. $(4r+7)(2r-3)$.
13. $(5r+4)(4r-2)$.
14. $(7r+6)(3r-2)$.
15. $(8r+9)(4r-3)$.
16. $(6s-5)(3s+2)$.
17. $(10s-7)(4s+1)$.
18. $(11s-12)(5s+4)$.
19. $(9s-12)(4s+5)$.
20. $(15s-20)(3s+2)$.
21. $(6ab+2)(3ab-5)$.
22. $(7mn-4)(6mn-3)$.
23. $(8r^2-3)(9r^2+4)$
24. $(13x^3+7)(5x^3-3)$.
25. $(6p^2-7)(4p^2+3)$.
26. $(11m^2-4)(5m^2+6)$.
27. $(9c^3-2)(6c^3+1)$.
28. $(10n^3+3)(15n^3-4)$.
29. $(14x^2-5)(2x^2+1)$.
30. $(16t^3-9)(3t^3+2)$.
31. $(14rs-9)(5rs+3)$.
32. $(2a-3b)(4a+5b)$.
33. $(6m+5s)(7m-4s)$.
34. $(8t-3x)(9t-x)$.
35. $(x-5y)(x+5y)$.
36. $(2x-3y)(2x+3y)$.
37. $(2m-5n)(2m-5n)$.
38. $(3t+4n)(3t+4n)$.
39. $(9ab-4c)(3ab+5c)$.
40. $(6xy-7z)(5xy+6z)$.
41. $(7r^2+8s)(8r^2-9s)$.
42. $(10x^3-11y^3)(11x^3+12y^3)$.
43. $(9mn^2+4)(6mn^2-3)$.
44. $(2+9x)(3+2x)$.
45. $(5-7t^2)(6+9t^2)$.
46. $(8+3z^3)(10-4z^3)$.
47. $(7-6xy)(5+6xy)$
48. $(9-11xy^2)(4-xy^2)$.
49. $(10a^2b-6c)(9a^2b-2c)$.
50. $(12x^2+7y)(5x^2-4y)$.
51. $(9t^2-5x^2)(7t^2+5x^2)$.
52. $(8m^2-9n)(4m^2+5n)$.

53. $(7x^2+3y^2)(5x^2-2y^2)$.

54. $(11c^2-5d)(4c^2-5d)$.

55. $(15p^2+2q)(9p^2-2q)$.

56. $(20a-7b)(8a+3b)$.

57. $(15m-2n)(6m+5n)$.

58. $(6m+\frac{1}{2})(4m+\frac{1}{3})$.

59. $(8x-\frac{2}{3}y)(9x-\frac{1}{2}y)$.

60. $(12a+\frac{2}{3}b)(10a-\frac{2}{3}b)$.

103. Factoring Trinomials of the Form mx^2+nx+p. The product of two binomials like $(2x+3)$ and $(3x+5)$ is a binomial of the *form* mx^2+nx+p

This means that there is a term containing the second power of x, one containing the first power (as a rule), and one free from x

The following discussion shows how to factor trinomials of the form mx^2+nx+p, when they are factorable.

DEVELOPMENT 1. Find the products:

(*a*) $(2x+6)(3x+2)$. (*b*) $(3x-5)(4x+7)$.

2. How do you obtain the "middle term" of the product?

3. Factor $12x^2+23x+5$

SOLUTION: 1. The first terms of the binomials might be $2x$ and $6x$, for their product is $12x^2$. Place them in parentheses thus: $(2x\quad)(6x\quad)$.

2. The second terms of the binomials are both positive since 5 and $23x$ are positive. Place the factors $+5$ and $+1$ in the parentheses and note the middle term which results.

(*a*) $(2x+5)(6x+1)$; middle term, $+32x$. Incorrect.

(*b*) $(2x+1)(6x+5)$; middle term, $+16x$. Incorrect

3. Step 2 shows that the factors $2x$ and $6x$ for $12x^2$ are incorrect. Try $3x$ and $4x$ for $12x^2$, thus: $(3x\quad)(4x\quad)$.

(*a*) $(3x+1)(4x+5)$; middle term, $+19x$. Incorrect

(*b*) $(3x+5)(4x+1)$; middle term, $+23x$. Correct.

CHECK: $(3x+5)(4x+1)=12x^2+23x+5$

NOTE. This may seem a long process at first, but practice soon develops such skill that most of the trial of factors can be done mentally.

EXERCISE 64

Factor:

1. $3x^2+5x+2.$
2. $3a^2+7a+2.$
3. $5m^2+7m+2.$
4. $7y^2+9y+2.$
5. $6x^2+7x+2.$
6. $12t^2+13t+3.$
7. $7a^2+10a+3.$
8. $6t^2+13t+6.$
9. $12w^2-17w+6.$

SOLUTION: 1. Since 6 is positive, its factors must have like signs; and, since -17 is negative, the cross products must both be negative. Therefore the factors of 6 must both be negative.

2. To get $12w^2$, use $(2w \quad)(6w \quad)$.

To get $+6$, try -2 and -3.

(*a*) $(2w-3)(6w-2)$; middle term, $-22w$. Incorrect.

(*b*) $(2w-2)(6w-3)$; middle term, $-18w$. Incorrect.

3. To get $12w^2$, use $(3w \quad)(4w \quad)$.

To get $+6$, try -2 and -3.

(*a*) $(3w-2)(4w-3)$; middle term, $-17w$. Correct.

CHECK: Does $(3w-2)(4w-3)=12w^2-17w+6$? Yes.

NOTE. In step 2, the factors 3 and 2 were used in both ways, as in (*a*) and (*b*); usually this is a wise plan, although in this case an explanation could be given to show that it was unnecessary.

10. $2w^2-11w+5.$
11. $2w^2-7w+3.$
12. $3b^2-19b+6.$
13. $6m^2-7m+2.$
14. $2w^2-9w+4.$
15. $6m^2-11m+3.$
16. $8a^2-22a+15.$
17. $5c^2-23c+12.$
18. Factor $15x^2+14x-8.$

SOLUTION: 1. The factors of -8 must have unlike signs. Arrange the signs so that the cross product of greater absolute value is positive.

2. For $15x^2$, try $(3x \quad)(5x \quad)$. For 8, try 2 and 4.

(*a*) $(3x-2)(5x+4)$; middle term, $+2x$. Incorrect.

The sign of 4 is made $+$, because $4 \cdot 3$ is greater than $2 \cdot 5$.

(*b*) $(3x+4)(5x-2)$; middle term, $+14x$. Correct.

The sign of 4 is made $+$, because $4 \cdot 5$ is greater than $3 \cdot 2$.

CHECK: Does $(3x+4)(5x-2)=15x^2+14x-8$? Yes.

19. Factor $24m^2 - m - 10$.

Solution: 1. The factors of -10 must have unlike signs. The cross product of greater absolute value must be negative.

2. For $24m^2$, try $(6m \quad)(4m \quad)$. For 10, try 5 and 2.

(*a*) $(6m + 2)(4m - 5)$; middle term, $-22m$. Incorrect.

(*b*) $(6m - 5)(4m + 2)$; middle term, $-8m$. Incorrect.

3. For $24m^2$, try $(3m \quad)(8m \quad)$.

(*a*) $(3m - 5)(8m + 2)$; middle term, $-34m$. Incorrect.

(*b*) $(3m - 2)(8m + 5)$; middle term, $-m$. Correct.

Check: Does $(3m - 2)(8m + 5) = 24m^2 - m - 10$? Yes.

Note. In this last step, for example, after placing 2 and 5 in the parentheses, we see that 2×8 is greater than 3×5; therefore make 2 negative and 5, positive.

20. $2x^2 - 3x - 5$.

21. $3m^2 + 4m - 7$.

22. $5t^2 - 2t - 7$.

23. $7r^2 + 4r - 11$.

24. $6s^2 - 7s - 5$.

25. $12s^2 + 5s - 3$.

26. $2x^2 + x - 15$.

27. $9r^2 - 6r - 8$.

28. $15c^2 - 4c - 3$.

29. $21A^2 + 2A - 8$.

30. $5y^2 + 16y + 3$.

31. $18x^2 - 3x - 10$.

32. $4x^2 - 24x + 35$.

33. $10x^2 - 13x - 30$.

34. $3z^2 + 22z + 7$.

35. $18m^2 + 17m - 15$.

36. $6x^2 + 31x + 35$.

37. $14m^2n^2 - 31mn - 10$.

38. $4r^2s^2 + 4rs - 15$.

39. $12x^2 + 17x + 6$.

40. $24m^2 - 18m - 15$.

41. $12t^2 + 13tx - 35x^2$.

42. $2w^2 - 3wr - 20r^2$.

43. $15y^2 + 19yz + 6z^2$.

44. $10n^2 + 9nt - 9t^2$.

45. $9b^2 + 3bc - 56c^2$.

46. $15a^2 - 26ab - 21b^2$.

47. $7x^2 - 26xy - 8y^2$.

48. $15a^2 + 29ab + 12b^2$.

49. $21x^2 - 29xy - 10y^2$.

50. $6r^2 - 25rs + 25s^2$.

51. $4c^2 - 8cd - 21d^2$.

52. $9d^4 - 6d^2 - 35$.

53. $8p^6 + 18p^3 - 35$.

54. $10m^4 + 19m^2n - 15n^2$.

55. $14v^2 + 29v - 15$.

56. $15k^2x - 16kx - 15x$.

(Remove the monomial first.)

57. $35at^2 + at - 12a$.

58. $24x^2r^3 - 7xyr^3 - 6y^2r^3$.

59. $55m^2x^5 + 2mnx^5 - 21n^2x^5$.

60. $24c^2d^3 + 2cd^3 - 15d^3$.

EXERCISE 65

Perform the various steps of the following indicated multiplications mentally:

1. $3a(2a+5)(a-4)$.

SOLUTION: $3a(2a+5)(a-4) = 3a(2a^2 - 3a - 20)$
$= 6a^3 - 9a^2 - 60a.$

2. $2m(3m-1)(2m+6)$.

3. $4ab(5x-2)(5x+2)$.

4. $7(3c+5d)(3c-4d)$.

5. $5(2r-5s)^2$.

6. $-3(6a-5)(2a+3)$.

7. $-2(7r-4s)(2r+3s)$

Solve the following equations, performing mentally all of the steps of the solution:

8. $2(3x-2)(x+4) - (6x+5)(x-3) = 65$.

SOLUTION: 1. $2(3x-2)(x+4) - (6x+5)(x-3) = 65$.

2. $2(3x^2 + 10x - 8) - (6x^2 - 13x - 15) = 65$.

3. Complete this example by removing parentheses, combining terms, etc.

9. $(5x-4)(3x+4) - 3(5x+6)(x-7) = 10(9x+15)$.

10. $(4r-5)(3r+7) - 2(r+1)(6r-7) = 3(2r+3)$.

Find *all* of the prime factors in the following; remember to remove the monomial factor first:

11. $24m^2a + 18mna - 15n^2a$.

12. $18bc^2 - 2bx^2$.

13. $12x^4y - 8x^3y^2 - 4x^2y^3$.

14. $15c^3d - 10c^2d - 25cd$.

15. $18k^2r^2 - 60klr^2 + 50l^2r^2$.

16. $48cd^2 + 120cde + 75ce^2$.

← My goodness →

17. $39\,m^2nx^2 + 20\,m^2nxy - 4\,m^2ny^2$.

18. $3\,x^4 - 17\,x^3 - 56\,x^2$.

19. $4\,x^4y^2 + 2\,x^3y^3 - 7\,x^3y^2$.

20. $45\,r^4s - 80\,r^2s^5$.

CASE VI

Type Form: $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

104. Factoring the Difference of Two Cubes.

DEVELOPMENT. 1. Divide $x^3 - 27$ by $x - 3$; by $x + 3$.

2. Divide $x^3 - y^3$ by $x - y$; by $x + y$.

3. Write the cube of a; the cube of b; the difference of these cubes.

4. Examine the results in 1 and 2; by what do you think you can divide $a^3 - b^3$ and get an exact quotient?

5. Prove by division the following fact:

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

Hence,
$$a^3 - b^3 = (a-b)(a^2+ab+b^2).$$

Rule.—1. The difference of the cubes of two numbers may be divided exactly by the difference of the numbers.

2. The quotient is the square of the first number, plus the product of the two numbers, plus the square of the second number.

EXAMPLE 1. Find a divisor of $8\,x^3 - 27$; find the quotient.

SOLUTION: 1. $8\,x^3 - 27 = (2\,x)^3 - 3^3$.

2. ∴ It is the difference of the cubes of $2\,x$ and 3, and can be divided exactly by $(2\,x - 3)$.

3. ∴ The quotient is $(2\,x)^2 + (2\,x)(3) + (3)^2$, or

$$4\,x^2 + 6\,x + 9.$$

Check by substitution or by multiplication.

EXAMPLE 2. Factor $8\,x^6 - 64\,y^3$.

SOLUTION: 1. $8\,x^6 - 64\,y^3 = (2\,x^2)^3 - (4\,y)^3$

2. $= (2\,x^2 - 4\,y)\{(2\,x^2)^2 + (2\,x^2)(4\,y) + (4\,y)^2\}$

3. $= (2\,x^2 - 4\,y)\{4\,x^4 + 8\,x^2y + 16\,y^2\}$.

NOTE. The middle term is *not* twice the product of the first and second numbers; try not to confuse this type with the one in § 96.

(20 min

EXERCISE 66

Factor:

1. $a^3 - 8$. 8 + w
2. $b^3 - 64$. 64 + v
3. $c^3 - 125$. 125 + c
4. $c^3 - 8d^3$.
5. $m^6 - 64n^3$
6. $r^6 - 8$.
7. $s^9 - 125$.
8. $x^{12} - 64y^3$.
9. $8b^3 - a^3$.
10. $64m^3 - 1$.
11. $8x^6 - 125y^3$.
12. $3x^3 - 24y^3$. (Remove monomial first.)
13. $40a^3 - 5b^3$. 8 27 216
14. $24x^6 - 81y^9$. 125
15. $4a - 500ax^6$.
16. $320m^3 - 5n^6$.
17. $x^3 - \frac{1}{8}$.
18. $y^3 - \frac{8}{27}$.
19. $z^3 - \frac{t^3}{125}$.

What is the quotient of:

20. $(x^3 - a^3) \div (x - a)$?
21. $(r^3 - x^6) \div (r - x^2)$?
22. $(27x^3y^3 - 1) \div (3xy - 1)$?
23. $(64a^6 - b^6) \div (4a^2 - b^2)$?
24. $(8 - x^9) \div (2 - x^3)$?
25. $(m^{12} - 27) \div (m^4 - 3)$?
26. $(27w^6 - 64v^3) \div (3w^2 - 4v)$?
27. $(1 - 125m^6) \div (1 - 5m^2)$?

CASE VII

Type Form: $(a^3 + b^3) = (a + b)(a^2 - ab + b^2)$.

105. Factoring the Sum of Two Cubes.

DEVELOPMENT. 1. Divide $(x^3 + 27)$ by $(x + 3)$; by $x - 3$.

2. Write the cube of m; of n; the sum of these results. Write the sum of m and n.

3. Divide the sum of the cubes of m and n, by the sum of m and n; also by their difference.

4. Prove by division the following fact:

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

Hence $$(a^3 + b^3) = (a + b)(a^2 - ab + b^2).$$

Rule 1. — The sum of the cubes of two numbers may be divided exactly by the sum of the two numbers.

2. The quotient is the square of the first number, minus the product of the two numbers, plus the square of the second number.

Example 1. Find an exact divisor of x^6+8; find the quotient.

Solution: $x^6+8=(x^2)^3+(2)^3$.

Since it is the sum of the cubes of two numbers, it may be divided exactly by the sum of the two numbers (x^2+2). (Rule 1.)

The quotient is $(x^2)^2-(x^2)\cdot(2)+(2)^2$ or x^4-2x+4. (Rule 2.)

Note. The middle term is *not twice* the product of the two numbers.

Example 2. Find the prime factors of $3mx^3+81my^6$.

Solution:
$$\begin{aligned}3mx^3+81my^6&=3m(x^3+27y^6).\\&=3m[(x)^3+(3y^2)^3]\\&=3m[x+3y^2][x^2-(3y^2)\cdot x+(3y^2)^2]\\&=3m(x+3y^2)(x^2-3xy^2+9y^4).\end{aligned}$$

EXERCISE 67

Change the sign in each example of Exercise 66 from minus to plus. This will give 27 examples.

106. Summary. In this chapter, seven special forms of algebraic expressions have been considered. These type forms are collected here for reference:

I. $a(x+y+z+\ldots)=ax+ay+az+\ldots.$
II. $(a\pm b)^2=a^2\pm 2ab+b^2.$
III. $(a+b)(a-b)=a^2-b^2.$
IV. $(x+a)(x+b)=x^2+(a+b)x+ab.$
V. $(ax+b)(cx+d)=acx^2+(ad+bc)x+bd.$
VI. $(a-b)(a^2+ab+b^2)=a^3-b^3.$
VII. $(a+b)(a^2-ab+b^2)=a^3+b^3.$

Familiarity with these forms makes it possible:

(*a*) to perform many multiplications in algebra and arithmetic mentally;

(*b*) to perform many divisions mentally;

(*c*) to factor many algebraic expressions.

These are the more important forms. Others are discussed in a later chapter. (XVI.)

Rule. — **To perform an indicated multiplication of two or more factors by means of the type forms:**

1. First find the product of all binomial and polynomial factors.

2. Then multiply by the monomial factor.

Example. Find the product $3x(2x + y)(2x - y)(4x^2 + y^2)$.

Solution:
$$\begin{aligned} &3x(2x + y)(2x - y)(4x + y^2)\\ &= 3x \cdot (4x^2 - y^2)(4x^2 + y^2)\\ &= 3x \cdot (16x^4 - y^4)\\ &= 48x^5 - 3xy^4. \end{aligned}$$

Rule. — **To find the prime factors of a given expression:**

1. First remove any monomial factor of the expression.

2. Factor the resulting polynomial factor or factors by the proper methods, until all of the prime factors have been found.

Example. Factor $3ax^6 - 21ax^3 - 24a$.

Solution:
$$\begin{aligned} &3ax^6 - 21ax^3 - 24a\\ &= 3a(x^6 - 7x^3 - 8)\\ &= 3a(x^3 - 8)(x^3 + 1)\\ &= 3a(x - 2)(x^2 + 2x + 4)(x + 1)(x^2 - x + 1). \end{aligned}$$

EXERCISE 68

MISCELLANEOUS EXAMPLES

Expand the following expressions:

1. $(3x + 2y)(4x - 3y)$.

2. $(\frac{3}{11}x - 6y)(\frac{3}{11}x + 6y)$.

3. $(a^4 + a^2 + 1)(a^2 - 1)$.

4. $(2a - b^2)^2$. $4a^2 - 4ab^2 + b^4$

5. $(7a^4 + b^2)(3a^2 - 8b^2)$.

6. $(a + b)(a - b)(a^2 + b^2)$.

7. $(9x - 2y)(x + y)$.

8. $3x(a - b)(a + b)$.

9. $(5a - 4b^2)(5a - 6b^2)$.

10. $3m(m^4 - 4m^2n^2 + 2n^4)$.

11. $7n(n^3+1)(n^3-1)$.
12. $3a(a+1)(a-1)$.
13. $(a^2-12y)(a^2+9y)$.
14. $(x+y)(x^2-xy+y^2)(x^3-y^3)$.
15. $(x-\frac{3}{4}y)^2$.
16. $(m-\frac{1}{2})(m-\frac{1}{4})$.
17. $2x(3x+1)(x-4)$.
18. $(B^2-4AC)(B^2-6AC)$.
19. $a(a^3+3)(a^3-3)$.
20. $5(a-b)(2a+3b)$.
21. $3y(2-y)(5+3y)$.
22. $7(3m+1)(m-6)$.
23. $(x^3+y^2)(x^3-y^2)$.
24. $(xy-4)(xy+16)$.
25. $2(x-4y)^2$.
26. $(x+1)(x-1)(x^2-3)$.
27. $(x+\frac{1}{2})(x^2-\frac{1}{2}x+\frac{1}{4})$.
28. $(\frac{2}{5}x^2-\frac{3}{4})(\frac{2}{5}x^2+\frac{3}{4})$.
29. $(2n-\frac{1}{2})^2$.
30. $(x-\frac{3}{4})(x+\frac{5}{8})$.

Find the prime factors of the following:

31. $5x^2-40x+80$
32. ax^6+6ax^3+9a.
33. $x^2y^2-5abxy+6a^2b^2$.
34. $3x^2-60xy-288y^2$.
35. $c^2+44c-45$.
36. c^3-64d^3.
37. $3x^3+24y^3$.
38. $m^4+23m^2n+132n^2$.
39. a^4-18a^2+77
40. $x^6-3x^3y-108y^2$.
41. $4x^2-28xy+49y^2$.
42. $169m^2r-n^2t^2r$.
43. $a^2+17a-38$.
44. $3c^2+132c-135$.
45. x^3-y^3.
46. $2d^2-26d-136$.
47. $125x^6-8y^3$.
48. $8a^2-5a-3$.
49. $169a^2+78ab+9b^2$.
50. $30m^2-47m-5$.
51. $24x^3m+81y^3m$.
52. a^6+14a^3+49.
53. $x^3y-216y^4$.
54. $3k^2+33k+72$.
55. $9x^2-4xy-13y^2$.
56. $3x^6-3y^6$.
57. a^4-256
58. $3a^3-108a$.
59. $c^2d^2+9cd-52$.
60. $5x^{12}-5y^{12}$.
61. $25r^2+60rs+36s^2$.
62. $x^2+\frac{2}{9}x+\frac{1}{81}$.

63. $196\,y^2-\frac{25}{169}\,a^4$.	67. $x^2z^2-20\,xz-69$.
64. $-25\,x+x^2+100$.	68. $18-19\,c+c^2$.
65. $300\,m^4-243\,x^2$.	69. $b-16+15\,b^2$.
66. $y^2+12\,y-108$.	70. $-50\,a+a^2+49$.

107. The skill acquired in this chapter in performing some multiplications mentally is of use in solving certain equations.

Example. Solve the equation:

$$6(4-x)^2-5(2\,x+7)(x-2)=22-(2\,x+3)^2.$$

Solution: Perform the operations mentally.

1. $6(4-x)^2-5(2\,x+7)(x-2)=22-(2\,x+3)^2.$
2. $\therefore 6(16-8\,x+x^2)-5(2\,x^2+3\,x-14)=22-(4\,x^2+12\,x+9).$
3. $\therefore 96-48\,x+6\,x^2-10\,x^2-15\,x+70=22-4\,x^2-12\,x-9.$
4. $\therefore 166-63\,x-4\,x^2=13-12\,x-4\,x^2.$
5. $\therefore -63\,x+12\,x=13-166.$
6. $\therefore -51\,x=-153.$
7. $\therefore x=3.$

Check: Does $6\overset{1}{\cancel{(4-3)}}{}^2-5\overset{13}{\cancel{(6+7)}}\overset{1}{\cancel{(3-2)}}=22-\overset{81}{\cancel{(6+3)}}{}^2$?

Does $6-65=22-81$?

Does $-59=-59$? Yes.

EXERCISE 69

Solve the following examples performing all of the work mentally:

1. $(5\,x+7)(3\,x-8)=(5\,x+4)(3\,x-5)$.
2. $(4\,m-7)^2=(2\,m-5)(8\,m+3)-2$.
3. $(5-3\,r)(3+4\,r)-(7+3\,r)(1-4\,r)=-28$.
4. $(1-3\,p)^2-(p+5)^2=4(p+1)(2\,p-3)$.
5. $2\,t(t+7)-(t-5)^2=(t+3)(t-11)$.
6. $(3\,x+5)(2\,x-3)-6(x-2)(x+13)=11$.
7. $(7\,a-2)(3\,a+5)-(4\,a+3)^2=5\,a(a+2)+1$.

8. $(5n-6)(5n+6)-5(2n-3)^2=5(n+12)(n-1)$.

9. $3(4a-5)^2-(6a-1)^2=7+(2a-9)(6a-11)$.

10. $(2x-7)^2-(5x-2)^2+3x(7x+5)=-4$.

11. Find two numbers whose difference is 6, and the difference of whose squares is 120.

12. Find two numbers whose sum is 13, and the difference of whose squares is 65.

13. Divide the number 20 into 2 parts such that the square of one exceeds the square of the other by 40.

14. Find four consecutive numbers such that the product of the first and third shall be less than the product of the second and fourth by 9.

15. Find two consecutive numbers such that the difference of their squares, plus 5 times the greater number, exceeds 4 times the less number by 27.

16. Find two consecutive numbers such that the sum of their squares exceeds twice the square of the smaller by 251.

17. One man travels a certain distance in as many hours as he travels miles per hour; another man travels the same distance in two hours less time by going three miles per hour faster. What was the rate and time of the first man?

EXERCISE 70

Problems about Area

1. Express the area of each of the following figures, assuming for each an altitude of 20 inches and a base of 30 inches:

(*a*) A rectangle (§ 17). (*b*) A triangle (§ 17).
(*c*) A parallelogram (§ 17).

2. Express the same areas if the altitude is $2x$ and the base $(x-5)$.

3. Express the same areas if the base is y and the altitude exceeds the base by 4.

4. The side of a square is s inches.

(a) Express the area of the square

(b) Express the dimensions of a rectangle if its base is 4 inches more and if its altitude is 3 inches less than the side of the square.

(c) Express the area of the rectangle.

(d) Form an equation expressing the fact that the area of the rectangle exceeds the area of the square by 50 square inches.

5. The base of a rectangle exceeds twice its altitude by 5 inches; the base of a triangle exceeds the base of the rectangle by 4 inches, and its altitude exceeds the altitude of the rectangle by 3 inches. Let a represent the altitude of the rectangle.

(a) Express the dimensions and area of the rectangle.

(b) Express the dimensions and area of the triangle.

(c) Form an equation expressing the fact that the area of the rectangle exceeds the area of the triangle by 25 square inches.

Equations

6. The base of a certain rectangle exceeds its altitude by 8 inches. If the base and altitude are both decreased by 2 inches, the old area exceeds the new by 36 square inches. Find the dimensions of the rectangle.

7. The base of a rectangle is 9 feet more and the altitude is 8 feet less than the side of a square. The area of the rectangle exceeds the area of the square by 15 square feet. Find the dimensions of the rectangle.

8. A man planned a house whose length exceeded its width by 10 feet. He found that it would be too expensive to build the house as planned, so he decided to decrease both

dimensions 5 feet. He found that this made a difference of 425 square feet in the area covered by the house. What were the original and the new dimensions?

9. The main shaft of Washington's Monument is square at the bottom and top. The side of the lower square exceeds the side of the upper square by 21 feet. The area of the lower square exceeds the area of the upper square by 1869 square feet. Find the dimensions of the two squares.

10. A man planned to set out an apple orchard with two more trees to each row than he had rows, but found that that plan left one tree over. He found that if he decreased the number of rows by 3, and increased the number of trees per row by 5, he used all of his trees. How many trees had he?

QUADRATIC EQUATIONS SOLVED BY FACTORING

108. Not all equations are simple or first degree equations. (§ 77.)

EXAMPLE. Find the number whose square exceeds the number itself by 6.

SOLUTION: 1. Let $n =$ the number.
2. Then $n^2 =$ the square of the number,
3. and $\therefore n^2 = n + 6.$
4. $\therefore n^2 - n - 6 = 0.$
5. Factoring: $(n - 3)(n + 2) = 0.$
6. If $(n - 3) = 0$, then $0 \cdot (n + 2)$ would also equal zero.
$n - 3 = 0$, if $n = 3$.
7. If $(n + 2) = 0$, then $(n - 3) \cdot 0$ would also equal zero.
$n + 2 = 0$, if $n = -2$.
8. These values of n, $+3$ and -2, should satisfy equation 4.

CHECK: If $n = 3$, does $(3)^2 - 3 - 6 = 9 - 3 - 6 = 0$? Yes.
If $n = -2$, does $(-2)^2 - (-2) - 6 = 4 + 2 - 6 = 0$? Yes.

9. Moreover, both of these numbers satisfy the conditions of the problem:

3^2 is 9; 9 exceeds 3 by 6.
$(-2)^2$ is 4; 4 exceeds -2 by 6.

109. An equation like $n^2 - n - 6 = 0$ is called a **Quadratic Equation** or an **Equation of the Second Degree.**

Other examples are: $4x^2 - 9 = 0$,

and $$3y^2 - \frac{2y}{5} = \frac{5}{3}y + 9.$$

Notice that the equation has only one unknown; that this unknown does not appear in the denominator of any fraction; that it does appear with exponent 2; that it may or may not appear with exponent 1.

Every quadratic equation has two roots, just as the equation in § 108.

110. Solution of Equations by Factoring depends upon the following numerical fact.

If *one* of *the factors* of a *product is zero,* the *value of the product is also zero.*

Thus, $3 \times 0 = 0$; $(-5) \times 0 = 0$; $2 \times 0 \times (-3) = 0 \times (-3) = 0$.

EXAMPLE 1. Solve the equation $4x^2 - 9 = 0$.

SOLUTION: 1. Factor: $(2x - 3)(2x + 3) = 0$.

2. If $2x - 3 = 0$, then $(2x - 3)(2x + 3) = 0$.
$2x - 3 = 0$, if $2x = 3$ or $x = +\frac{3}{2}$.

3. If $2x + 3 = 0$, then $(2x - 3)(2x + 3) = 0$.
$2x + 3 = 0$, if $2x = -3$, or $x = -\frac{3}{2}$.

4. The roots of the equation are $+\frac{3}{2}$ and $-\frac{3}{2}$.

5. CHECK: Does $4(\frac{3}{2})^2 - 9 = 0$?

Does $\overset{1}{\cancel{4}} \cdot \frac{9}{\cancel{4}} - 9 = 0$? *i.e.* $9 - 9 = 0$? Yes.

Does $4(-\frac{3}{2})^2 - 9 = 0$?

Does $\overset{1}{\cancel{4}} \cdot \frac{9}{\cancel{4}} - 9 = 0$? *i.e.* $9 - 9 = 0$? Yes.

Rule. — To solve an equation by factoring:

1. Transpose all terms to the left member.

2. Factor the left member completely.

3. Set each factor equal to zero, and solve the resulting equations.

4. The roots obtained in step 3, are the roots of the given equation. Check by substitution in the given equation.

Example 2. Solve the equation $\frac{m^2}{3} - \frac{m}{2} = \frac{35}{6}$.

Solution: 1. M_6:* $2m^2 - 3m = 35$.

2. S_{35}: $2m^2 - 3m - 35 = 0$.

3. Factor: $(2m + 7)(m - 5) = 0$.

4. $2m + 7 = 0$ if $m = -\frac{7}{2}$.

$m - 5 = 0$ if $m = +5$.

5. The roots of the equation are $+5$ and $-\frac{7}{2}$.

Check by substitution.

EXERCISE 71

Solve by factoring the following quadratic equations.

1. $x^2 - 12x + 32 = 0$.
2. $y^2 - 6y = 55$.
3. $n^2 = 63 - 2n$.
4. $m^2 - 18m = -72$.
5. $x^2 - 21x + 110 = 0$.
6. $c^2 = 2 - c$.
7. $d^2 - 7d + 6 = 0$.
8. $v^2 - 21 = 4v$.
9. $9m^2 - 4 = 0$.
10. $36x^2 - 25 = 0$.

here

11. $x^2 - 5x = 0$.
12. $5c^2 - 2c = 0$.
13. $8a^2 - 5a - 3 = 0$.
14. $4p^2 + 8p = 21$.
15. $24r^2 + 2r = 35$.
16. $4x^2 = 8x - 3$.
17. $3w^2 = -\frac{7}{2}w - 1$.
18. $\frac{3}{10}t^2 + \frac{13}{10}t = 1$.
19. $z^2 - \frac{10}{3}z + 1 = 0$.
20. $x^2 + \frac{5}{2}x + 1 = 0$.

111. A **Literal Equation** is one in which some or all of the known quantities are represented by letters; as,

$$2x + a = bx^2 - 10.$$

Example 1. Find two numbers whose difference is a, and whose product is 6 times the square of a.

Solution: 1. Let x = the larger number.

2. Then $x - a$ = the smaller number.

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* For the symbol M_6 see § 42.

3. $\therefore x(x-a)=6a^2.$

4. $\therefore x^2-ax-6a^2=0.$

5. $\therefore (x-3a)(x+2a)=0.$

6. $\therefore x=3a$ or $x=-2a$, the larger number.

CHECK: If $x=3a$, then $3a-a=2a$, the smaller number; and $3a\cdot 2a$ does equal $6a^2$.

If $x=-2a$, then $-2a-a=-3a$, the smaller number; and $-2a\cdot -3a$ does equal $+6a^2$.

Each solution is correct, no matter what a may be.

Thus, if a is 5, $3a$ is 15, $2a$ is 10, and 15×10 is 150; also $6\cdot a^2=150$.

EXAMPLE 2. Solve the equation $\frac{x^2}{2}-\frac{3cx}{2}=5c^2.$

SOLUTION: 1. $\frac{x^2}{2}-\frac{3cx}{2}=5c^2.$

2. M_2: $x^2-3cx-10c^2=0.$

3. Factoring: $(x-5c)(x+2c)=0.$

4. If $x-5c=0$, then $x=5c$.

5. If $x+2c=0$, then $x=-2c$.

Check by substitution.

EXERCISE 72

Solve the following equations for x:

1. $x^2+2ax-35a^2=0.$
2. $16x^2-9b^2=0.$
3. $x^2+23mx+130m^2=0.$
4. $25x^2=9c^2.$
5. $2x^2+7ax+3a^2=0.$
6. $7x^2-10bx+3b^2=0.$
7. $6x^2+7cx=5c^2.$
8. $13x^2=tx+14t^2.$
9. $9x^2=4xv+13v^2.$
10. $7x^2+41xk=6k^2.$
11. $\frac{x^2}{2}-ax-\frac{3a^2}{2}=0.$
12. $\frac{x^2}{4}=5p^2-2px.$
13. $\frac{x^2}{6}+6c^2=\frac{5cx}{2}.$
14. $\frac{x^2}{12}-\frac{4xn}{3}+4n^2=0.$

15. $x^2 - \frac{15\,bx}{2} = 4\,b^2.$

16. $6\,x^2 - \frac{47\,mx}{5} = m^2.$

17. $\frac{8\,x^2}{15} = \frac{ax}{3} + \frac{a^2}{5}.$

18. $\frac{x^2}{10} - \frac{7\,xt}{15} = \frac{t^2}{6}.$

19. $\frac{x^2}{2} - \frac{5\,rx}{6} = \frac{11\,r^2}{3}.$

20. $\frac{6\,x^2}{5} - \frac{4\,xs}{5} - \frac{3\,s^2}{2} = 0.$

112. When solving a problem whose solution leads to a quadratic equation, two sets of results are obtained. In some cases, both sets satisfy the conditions of the problem; in other cases, only one set satisfies the given conditions.

EXAMPLE 1. Find three consecutive odd numbers such that when the product of the first and third is increased by twice the square of the second, the sum equals 23.

SOLUTION: 1. Let $x =$ the smallest number.

2. $\therefore x + 2$ and $x + 4$ are the other two numbers.

(For example, if 3 is the smallest, $3 + 2$ or 5 and $3 + 4$ or 7 are the other two.)

3. $\therefore x(x + 4) + 2(x + 2)^2 = 23.$

$\therefore x^2 + 4\,x + 2(x^2 + 4\,x + 4) = 23.$

$\therefore 3\,x^2 + 12\,x + 8 = 23.$

$\therefore 3\,x^2 + 12\,x - 15 = 0,$ or $x^2 + 4\,x - 5 = 0.$

$\therefore (x + 5)(x - 1) = 0.$

$x = -5$ or $x = +1.$

4. When $x = -5$, $x + 2 = -3$, and $x + 4 = -1.$

When $x = +1$, $x + 2 = 3$, and $x + 4 = 5.$

The solutions are -5, -3, and -1; and 1, 3, and 5.

CHECK: $(-5)\cdot(-1) + 2(-3)^2 = 5 + 18 = 23.$ $1\cdot 5 + 2\cdot 3^2 = 5 + 18 = 23.$

In this case, both solutions are satisfactory.

EXAMPLE 2. Determine the base and altitude of a triangle when the area is 110 square inches and the base exceeds the altitude by 9 inches.

Solution: 1. Let $a =$ the number of inches in the altitude.

2. $\therefore a + 9 =$ the number of inches in the base.

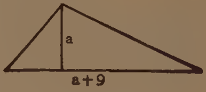
3. $\therefore \frac{a(a+9)}{2} =$ the area.

$$\therefore \frac{a(a+9)}{2} = 110.$$

$$\therefore a^2 + 9a - 220 = 0.$$

$$\therefore (a+20)(a-11) = 0.$$

$$\therefore a = 11, \text{ or } -20.$$



4. When $a = 11$, the base is $+20$ and the area is $(11 \times 20) \div 2$ or 110. This satisfies the conditions of the problem.

When $a = -20$, the result can have meaning only if we have triangles with negative altitude. In such cases we agree to take only the positive solution.

EXERCISE 73

✓ 1. Twice the square of a certain number equals the excess of 15 over the number. Find the number.

2. Find three consecutive numbers whose sum is equal to the product of the first two.

✓ 3. Divide 18 into two parts so that the sum of the squares of the parts shall be 170.

4. Find two numbers whose sum is 7 and the sum of whose cubes is 133.

✓ 5. Determine the base and altitude of a triangle such that the area shall be 15 square feet and the altitude shall be 7 feet less than the base.

6. Central Park in New York covers an area of about 800 acres. Its length exceeds its width by 2 miles. Find the dimensions of the park. (A square mile contains 640 acres.)

✓ 7. A merchant sold goods for $18.75 and lost as many per cent as the goods cost dollars. What was the cost?

8. The length of a certain rectangular farm is three times its width If its length should be increased by 20 rods, and its width by 8 rods, its area would be trebled. Of how many square rods does the farm consist?

9. The standard size city lot in parts of Chicago is five times as long as it is wide. The lots in parts of Indianapolis are 10 feet wider and 5 feet longer than those in Chicago. Three times the area of a Chicago lot exceeds twice the area of an Indianapolis lot by 275 square feet. Find the dimensions of the lots in both cities.

10. An architect who has made plans for a house with a base 30 × 42 feet finds that he must reduce the size By what equal amount must he reduce the two dimensions of the house in order to make the area of the new base 925 square feet?

IX. HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

113. A monomial is said to be **Rational** and **Integral** when it is either an arithmetical number, or a single literal number with unity for its exponent, or the product of two or more such numbers.

Thus, 3, a, $2a^3bc^2$ are rational and integral.

114. The **Degree** of a rational and integral monomial is the sum of the exponents of its literal factors.

Thus, a^4bc^3 is of the *eighth* degree.

115. A polynomial is said to be rational and integral when each term is rational and integral; as, $2a^2b - 3c + d^2$.

The degree of a rational and integral polynomial is the degree of its term of highest degree.

Thus, $2a^2b - 3c + d^2$ is of the *third* degree.

116. Recall the definition of prime factor of an expression (§ 87), and common factor (§ 11) of two or more expressions.

Thus the prime factors of:
(a) $6m^2(x^2 - 1)$ are 2, 3, m, m, $(x - 1)$, and $(x + 1)$.
(b) $9m^3(x^3 - 1)$ are 3, 3, m, m, m, $(x - 1)$, and $(x^2 + x + 1)$.
Common factors of (a) and (b) are 3, m, m, and $(x - 1)$.

EXERCISE 74

Select the common factors in the following sets of expressions:

1. $5 \cdot mn^2(x + 1)$.
   $3 \cdot 5 \cdot m^2n(x - 1)$.

2. $2 \cdot 3 \cdot x^3 \cdot y^2(x + 4)(x - 3)$.
   $2 \cdot 2 \cdot 3 \cdot x^4 \cdot y^2(x + 4)$.

3. $7 \cdot 5 \cdot r^3 \cdot s^2(x+3)(x-2)(x-1).$
$3 \cdot 7 \cdot 5 \cdot r^2 \cdot s(x-3)(x-2)(x-1).$

4. $6 \cdot 7 \cdot m^5n^4(x-a)(x^2+a^2).$
$6 \cdot 7 \cdot 7 \cdot m^7n^5(x+a)(x^2+a^2).$

5. $3 \cdot 5 \cdot 7 \cdot 2 \cdot 11.$
$2 \cdot 2 \cdot 3 \cdot 5 \cdot 13.$

6. $3 \cdot 7 \cdot 5 \cdot 2 \cdot 2 \cdot 2.$
$2 \cdot 2 \cdot 7 \cdot 11.$

117. The **Highest Common Factor (H. C. F.)** of two or more rational and integral expressions is the expression of highest degree (§ 114), with greatest numerical coefficient, which will divide each of them without a remainder.

EXAMPLE. Find the H. C. F. of

$$5\,m^3n^2(x-1)(x+2) \text{ and } 15\,m^4n(x+1)(x+2).$$

SOLUTION: 1. The greatest integer which will divide both expressions is 5.

2. The highest power of m which will divide both is m^3.
The highest power of n which will divide both is n.
The highest power of $(x+2)$ which will divide both is $(x+2)$.
Neither $(x-1)$ nor $(x+1)$ will divide both expressions.

3. $\therefore$ The H. C. F. is $5\,m^3n(x+2)$.

CHECK: $5\,m^3n^2(x-1)(x+2) \div 5\,m^3n(x+2) = n(x-1).$
$15\,m^4n(x+1)(x+2) \div 5\,m^3n(x+2) = 3\,m(x+1).$

Rule. — To find the H. C. F. of two or more expressions:

1. Find all of the prime factors of each expression.

2. Select the factors common to all of the given expressions, and give each the lowest exponent it has in any of the expressions.

3. Form the product of the common factors selected in step 2.

EXAMPLE. Find the H. C. F. of $68(m+n)^2(m-n)^4$ and $85(m+n)^3(m-n)$.

SOLUTION: 1. $68(m+n)^2(m-n)^4 = 2 \cdot 2 \cdot 17 \cdot (m+n)^2(m-n)^4.$
$85(m+n)^3(m-n) = 5 \cdot 17 \cdot (m+n)^3(m-n).$

2. $\therefore$ The H. C. F. $= 17(m+n)^2(m-n).$

5 - 3 - 7

EXERCISE 75

Find the highest common factor of:

1. 16 and 56.
2. $64a$ and $96b$.
3. $72x^2$ and $27xy$.
4. $5a^4b^2$ and $2a^3b^2$.
5. $20x^2y$ and $15xy^2$.
6. $14x^3y$, $21xy^4$, and $35x^2y^2$.
7. $15m^5n^3$, $45m^4n^4$, and $25mn^2$
8. $12x^2$, $18x^3y^2$, and $24x^5y$.
9. $16m^4n^2$, $56m^2n^4$, and $88m^3n^2$.
10. $18r^3s$, $27r^2s^3$, and $45rs^2$.
11. $(a+b)(a-b)$ and $2a(a+b)$.
12. $3x(x+y)(x-y)$ and $2(x+y)(x-y)(x^2+y^2)$.
13. $(x+1)^2(x-3)$ and $(x+1)^2(x+2)$.
14. $(r+s)^3(r-s)^2$ and $(r+s)^2(r-s)^3$.
15. $3(x-2y)(x+2y)$ and $6(x-2y)^2$.
16. $2a^3x+4a^2x^2+2ax^3$ and $3a^4x+3ax^4$.

SOLUTION: 1.
$$\begin{aligned}2a^3x+4a^2x^2+2ax^3 &= 2ax(a^2+2ax+x^2)\\ &= 2ax(a+x)(a+x).\\ 3a^4x+3ax^4 &= 3ax(a^3+x^3)\\ &= 3ax(a+x)(a^2-ax+x^2).\end{aligned}$$

2. The H. C. F. $= a \cdot x \cdot (a+x) = ax(a+x)$.

17. a^2-b^2 and $a^2-2ab+b^2$.
18. $x^2+2x-24$, $x^2-14x+40$, and $x^2-8x+16$.
19. $2r^2-7r+6$ and $6r^2-11r+3$.
20. x^3-27 and $x^2-11x+24$.
21. m^5-8m^2 and m^2+2m+4.
22. $6a^3b^2-15a^2b^3$ and $12a^4b+21a^3b^2$.
23. $3a^3+192$ and $a^2-7a-44$.
24. $3x^2-16xy+5y^2$ and $x^2+10xy-75y^2$.
25. $27a^3+8b^3$, $9a^2-4b^2$, and $9a^2+12ab+4b^2$.

LOWEST COMMON MULTIPLE

118. A **Multiple** of a number is any number which contains the given number as an exact divisor. Thus:

(*a*) Some multiples of 3 are 6, 9, 12, and 30.

(*b*) Some multiples of $(x + y)$ are $2(x + y)$ and $(x + y)(x - y)$.

119. One number may be a multiple of two or more different numbers. Thus:

(*a*) 24 is a multiple of 2, 3, 4, 6, 8, and of 12.

(*b*) $5a^2bc$ is a multiple of 5, a^2, b, c, and of a.

(*c*) $36m(x + y)(x - y)$ is a multiple of 3, m, $(x + y)$, and of $(x - y)$.

A **Common Multiple** of two or more numbers is a multiple of each of them; it can be divided exactly by each of them.

120. In arithmetic, it is necessary at times to find the smallest number which is a common multiple of two or more numbers. Thus, 30, 45, and 60 are all common multiples of 3 and 5; but the lowest common multiple of 3 and 5 is 15.

Similar necessity arises in algebra.

121. The **Lowest Common Multiple** of two or more rational and integral expressions is the expression of lowest degree (§ 114), with least numerical coefficient, which can be exactly divided by each of them.

EXAMPLE. Find the L. C. M. of $5a^5b^3$ and $7a^2b^4$.

SOLUTION: 1. The least number which will contain both 5 and 7 is 35.

2. The lowest power of a which will contain both a^5 and a^2 is a^5.

3. The lowest power of b which will contain both b^3 and b^4 is b^4.

4. The L. C. M. is $35a^5b^4$.

CHECK: Does $35a^5b^4$ contain each of the given numbers?

$$35a^5b^4 \div 5a^5b^3 = 7b; \quad 35a^5b^4 \div 7a^2b^4 = 5a^3.$$

Rule. — To find the L. C. M. of two or more expressions:

1. Find the prime factors of each of the expressions.

2. Select all of the different prime factors and give to each the highest exponent with which it occurs in any of the expressions.

3. Form the product of all of the factors selected in step 2.

EXAMPLE Find the L. C. M. of $25\,a^2b(a+b)^3(a-b)$ and $15\,a^3b^4(a+b)^2(a-b)^3$.

SOLUTION: 1. $25\,a^2b(a+b)^3(a-b) = 5^2\,a^2b(a+b)^3(a-b)$

$15\,a^3b^4(a+b)^2(a-b)^3 = 3\cdot 5\,a^3b^4(a+b)^2(a-b)^3$.

2. 3 occurs with 1 as its highest exponent.

5 occurs with 2 as its highest exponent.

a occurs with 3 as its highest exponent.

b occurs with 4 as its highest exponent.

$(a+b)$ occurs with 3 as its highest exponent.

$(a-b)$ occurs with 3 as its highest exponent.

3. The L. C. M. $= 3\cdot 5^2\,a^3b^4(a+b)^3(a-b)^3$.

CHECK: Does the L. C. M. contain each of the expressions?

$75\,a^3b^4(a+b)^3(a-b)^3 \div 25\,a^2b(a+b)^3(a-b) = 3\,ab^3(a-b)^2$.

$75\,a^3b^4(a+b)^3(a-b)^3 \div 15\,a^3b^4(a+b)^2(a-b)^3 = 5(a+b)$.

EXERCISE 76

Find the L. C. M. of the following and obtain the quotient when the L. C. M. is divided by each of the numbers:

1. 5 and 7.
2. 12 and 20.
3. 24 and 30.
4. 12 and 54.
5. 15, 21, and 33.
6. 20, 27, and 90.
7. $3\,ab$ and $7\,a^2b$.
8. $12\,x^2y^3$ and $48\,xy^4$.
9. $12\,m^3$ and $15\,mn^2$.
10. $24\,a^3b^4$ and $16\,a^2b^5$.
11. $14\,r^4s^2$ and $35\,rs^3$.
12. $15\,x^3y$, $30\,xy^2$, and $60\,x^2y^3$.
13. $5\,rm^2$, $15\,rm^3$, and $21\,r^2m$.
14. $24\,p^4$, $32\,p^3r$, and $12\,p^5$.
15. $32\,wv$, $16\,w^3v^2$, and $64\,w^4$.
16. $44\,xy$, $33\,yz$, and $12\,xz$.
17. $(a+b)(a-b)$ and $(a+b)^2$.
18. $2\,a(m+a)$ and $6\,a^2(m+a)(m-a)$.
19. $(x+3)(x-2)$ and $(x-2)(x-3)$.

20. $3(r+s)(r-s)$ and $2(r+s)(r-t)$.

21. $(a-4)(a-3)$ and $(a-4)(a-5)$.

22. $(1-x)^3$, $(1-x)(1+x)$, and $(1-x)^2$.

23. $(2-3x)^2$, $3(2-3x)(2+3x)$, and $(2+3x)^2$.

24. $(2-x)(3-x)$, $(3-x)(4-x)$, and $(4-x)(2-x)$.

25. $4x^2-4m^2$, $6x+6m$, and $3x^3-3m^3$.

SOLUTION: 1. $4x^2-4m^2=4(x^2-m^2)=4(x-m)(x+m)$.
$6x+6m=6(x+m)$.
$3x^3-3m^3=3(x^3-m^3)=3(x-m)(x^2+mx+m^2)$.

2. The L. C. M. $=12(x-m)(x+m)(x^2+mx+m^2)$.

3. L. C. M. $\div 4(x-m)(x+m)=3(x^2+mx+m^2)$.
L. C. M. $\div 6(x+m)=2(x-m)(x^2+mx+m^2)$.
L. C. M. $\div 3(x-m)(x^2+mx+m^2)=4(x+m)$.

found will 22

26. r^2-16 and $r^2+11r+28$.

27. $a^2+2ax+x^2$ and a^3+x^3.

28. $8x^3y-y$ and $22x^2z-9xz-z$.

29. $b^2-12b+35$, $b^2+2b-63$, and $b^2-3b-108$.

30. $4x^2-25$ and $4x^2-20x+25$.

31. $3m^2-6m-72$, $4m^2+8m-192$, and $2m^2-24m+72$.

32. $9n^2-27n+8$ and $3n^2-2n-16$.

33. $1+27x^3$ and $1-5x-24x^2$.

34. x^2+x-42 and x^2-x-30.

35. a^3-a and a^3-9a^2-10a.

X. FRACTIONS

122. The quotient of a divided by b is written $\frac{a}{b}$. The expression $\frac{a}{b}$ is called a **Fraction**; the dividend, a, is called the **Numerator**, and the divisor, b, is called the **Denominator** The numerator and denominator are called the **Terms** of the fraction. $\frac{a}{b}$ is read "a divided by b." The denominator, b, must never be zero (§ 64).

REDUCTION OF FRACTIONS TO LOWEST TERMS

123. A fraction is said to be in its **Lowest Terms** when its numerator and denominator have no common factor except unity. Thus:

(*a*) $\frac{2}{3}$, $\frac{a}{b}$, $\frac{x+y}{x-y}$ are in their lowest terms.

(*b*) $\frac{3}{12}$, $\frac{2a^2}{ab}$, $\frac{(x+y)^2}{(x^2-y^2)}$ are not in their lowest terms

124. To reduce a fraction to lowest terms, a principle, easily illustrated by arithmetical fractions, is used.

If, in the fraction $\frac{24}{8}$, both terms be divided by 4, a new fraction $\frac{6}{2}$ is obtained. But, since $\frac{24}{8} = 3$, and $\frac{6}{2} = 3$, therefore $\frac{24}{8} = \frac{6}{2}$. The *value* of the fraction is not changed; its *form* is changed.

Rule. — If the numerator and denominator of a fraction are both divided by the same number, the value of the fraction is not changed.

Example 1. Reduce to lowest terms $\frac{24a^3b^2c}{40a^2b^2c}$.

Solution: 1. $\frac{24a^3b^2c}{40a^2b^2c} = \frac{2^3 \cdot 3 \cdot a^3 \cdot b^2 \cdot c}{2^3 \cdot 5 \cdot a^2 \cdot b^2 \cdot c}$.

2. Divide both numerator and denominator successively by their common factors, 2^3, a^2, b^2, and c.

Then, $$\frac{24a^3b^2c}{40a^2b^2c}=\frac{\overset{1}{\cancel{2^3}}\cdot 3\cdot \overset{a}{\cancel{a^3}}\cdot\overset{1}{\cancel{b^2}}\cdot\overset{1}{\cancel{c}}}{\underset{1}{\cancel{2^3}}\cdot 5\cdot\underset{1}{\cancel{a^2}}\cdot\underset{1}{\cancel{b^2}}\cdot\underset{1}{\cancel{c}}}=\frac{3a}{5}.$$

Note 1. Dividing by all of the common factors is equivalent to dividing both numerator and denominator by their highest common factor.

Note 2. The process of removing a common factor from numerator and denominator of a fraction is called cancellation. It depends upon the rule in § 124.

Note 3. It is wise to write the quotients 1, a, etc., as they are obtained.

Example 2. Reduce to lowest terms $\dfrac{27a^3+64b^3}{9a^2+24ab+16b^2}$.

Solution: $$\frac{27a^3+64b^3}{9a^2+24ab+16b^2}=\frac{\overset{1}{\cancel{(3a+4b)}}(9a^2-12ab+16b^2)}{\underset{1}{\cancel{(3a+4b)}}(3a+4b)}.$$

$$=\frac{9a^2-12ab+16b^2}{(3a+4b)}.$$

Here, the numerator and denominator are both divided by $(3a+4b)$.

Check: These examples may be checked by substitution. It is important to remember that the original fraction and the simplified result are equal for all values of the literal numbers except such as make the denominator zero. (§ 122.)

Rule. — To reduce a fraction to lowest terms:

1. Express numerator and denominator in terms of their prime factors.

2. Divide both numerator and denominator by all of their common factors, *i.e.* by their H. C. F.

125. Errors in Reducing Fractions. One common error occurs in reducing fractions such as

$$\frac{3mn^2(x+y)}{3mn^2(x+y)}=\frac{\cancel{3}\cancel{m}\cancel{n^2}\cancel{(x+y)}}{\cancel{3}\cancel{m}\cancel{n^2}\cancel{(x+y)}}.$$

Pupils sometimes think that the result is 0, because all factors have been cancelled. If, as suggested, the quotients are indicated, this danger will be avoided; thus,

$$\frac{3mn^2(x+y)}{3mn^2(x+y)}=\frac{\overset{1}{\cancel{3}}\overset{1}{\cancel{m}}\overset{1}{\cancel{n^2}}\overset{1}{\cancel{(x+y)}}}{\underset{1}{\cancel{3}}\underset{1}{\cancel{m}}\underset{1}{\cancel{n^2}}\underset{1}{\cancel{(x+y)}}}=\frac{1}{1}=1.\ \textit{Ans.}$$

Another, and more common, error is illustrated in the *following faulty solution:*

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$$\frac{2a+b}{ab}=\frac{2\overset{1}{\cancel{a}}+\overset{1}{\cancel{b}}}{\underset{1}{\cancel{a}}\underset{1}{\cancel{b}}}=\frac{2+1}{1}=3.$$

Test it for $a=2,\ b=2$; $\frac{2a+b}{ab}=\frac{4+2}{4}=\frac{6}{4}=\frac{3}{2}$.

The error is in dividing one part of the numerator by a and another part of it by b. Neither a nor b is a factor of the numerator. They cannot be canceled in this problem. The fraction is already in its lowest terms.

EXERCISE 77

Reduce the following fractions to their lowest terms:

1. $\frac{12}{15}$.
2. $\frac{24}{36}$.
3. $\frac{5x^3y^4}{3xy^5}$.
4. $\frac{12m^5n^3}{42m^3n^5}$.
5. $\frac{45r^4st^2}{63r^3s^2t^2}$.
6. $\frac{65x^3y^4z}{40xy^4z}$.
7. $\frac{3ab^2c(x+y)}{2abc(x+y)}$.
8. $\frac{12(a-3)(a+3)}{4(a+3)}$.
9. $\frac{18(2r-s)^2(r+s)}{9(2r-s)(r+s)^2}$.
10. $\frac{a^2+7a+10}{a^2+4a-5}$.
11. $\frac{m^2+m-56}{m^2-m-42}$.
12. $\frac{x^2-9x+18}{3x^2+3x-36}$.
13. $\frac{a^2-11ab+28b^2}{a^2-14ab+49b^2}$.
14. $\frac{2x^2-2y^2}{5x^3-5y^3}$.
15. $\frac{3am^2-3an^2}{3m^2+6mn+3n^2}$.
16. $\frac{a^3+b^3}{a^2c-2abc-3b^2c}$.
17. $\frac{8x^3-125}{2ax^2+ax-15a}$.
18. $\frac{mx^2-mx-12m}{3x^2+13x+12}$.
19. $\frac{16r^4+4r^2+1}{64r^6-1}$.
20. $\frac{a^3-8a^2+12a}{5a^2-60a+180}$.

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21. $\dfrac{x^2-y^2}{ax^2-2\,axy+ay^2}$.

22. $\dfrac{a^3-64}{a^2m-16\,m}$.

23. $\dfrac{a^3y-8\,y}{(a^2-4)(a-1)}$.

24. $\dfrac{4\,m^2+16\,mn+15\,n^2}{6\,m^2-mn-15\,n^2}$.

25. $\dfrac{15\,x^4y+10\,x^3y^2}{6\,x^3y^4+4\,x^2y^5}$.

26. $\dfrac{3\,x^4-3\,y^4}{x^4+2\,x^2y^2+y^4}$.

126. Signs in Fractions. A fraction is an indicated quotient. Its sign is governed by the law of signs for division (§ 67). Thus:

1. $\dfrac{+12}{+4}=+3.$
2. $\dfrac{+12}{-4}=-3.$
3. $\dfrac{-12}{+4}=-3.$
4. $\dfrac{-12}{-4}=+3.$

From 1 and 2, it is clear that

$$\frac{+12}{+4}=-\frac{+12}{-4},$$

From 1 and 3, it is clear that

$$\frac{+12}{+4}=-\frac{-12}{+4},$$

since $+3=-(-3)$.

Rule. — If the sign of one term of a fraction is changed, the sign of the whole fraction must be changed.

From 1 and 4, it is clear that

$$\frac{+12}{+4}=\frac{-12}{-4},\text{ since both equal }+3.$$

Rule. — If the signs of both terms of a fraction are changed, the sign of the fraction must not be changed.

Changing the signs of an expression is accomplished by multiplying it by -1.

Example 1. Reduce to lowest terms:

$$\frac{x^2-9}{12+2\,x-2\,x^2}.$$

$-3x$ [illegible] 2 2

Solution: Multiply the denominator of the given fraction by -1, thus changing its signs; also, change the sign of the fraction.

$$\frac{x^2-9}{12+2x-2x^2}=-\frac{x^2-9}{2x^2-2x-12}=-\frac{\overset{1}{\cancel{(x-3)}}(x+3)}{2\underset{1}{\cancel{(x-3)}}(x+2)}=-\frac{x+3}{2(x+2)}.$$

Check. Let $x=1$. Then $\dfrac{x^2-9}{12+2x-2x^2}=\dfrac{1-9}{12+2-2}=\dfrac{-8}{12}=-\dfrac{2}{3}$;

also, $-\dfrac{x+3}{2(x+2)}=-\dfrac{1+3}{2(1+2)}=-\dfrac{4}{6}=-\dfrac{2}{3}$.

The value of x selected *must not make any* denominator zero (§ 64).

Example 2. Reduce to lowest terms:

$$-\frac{(9-m^2)}{m^2-7m+12}.$$

Solution: Multiply the numerator by -1, thus changing its signs; also change the sign of the fraction.

1. $-\dfrac{9-m^2}{m^2-7m+12}=+\dfrac{m^2-9}{m^2-7m+12}$

2. $=\dfrac{\overset{1}{\cancel{(m-3)}}(m+3)}{(m-4)\underset{1}{\cancel{(m-3)}}}=\dfrac{(m+3)}{(m-4)}.$

EXERCISE 78

Reduce to lowest terms:

1. $\dfrac{4y^2-x^2}{2x^2-7xy+6y^2}.$

2. $\dfrac{3x-3y}{y^2+xy-2x^2}.$

3. $\dfrac{a^2-16}{24-2a-a^2}.$

4. $\dfrac{3m^2-3n^2}{n^2+4mn-5m^2}.$

5. $\dfrac{x^2+x-6}{15+2x-x^2}.$

6. $-\dfrac{27-t^3}{t^2-9}.$

7. $-\dfrac{x^2-16}{64-x^3}.$

8. $\dfrac{ar^2-as^2}{2as^2+4ars-6ar^2}.$

9. $\dfrac{3c^3-6c^2d+3cd^2}{5d^2-5c^2}.$

10. $\dfrac{18mx^2-8my^2}{24ny^3-81nx^3}.$

TO REDUCE A FRACTION TO AN INTEGRAL OR MIXED EXPRESSION

127. An **Integral Expression** is an expression which has no fractional *literal* part; as, $a^2 - 2ab + b^2$, or $\frac{2}{3}ab^2$.

An integral expression may be considered a fraction whose denominator is 1; thus, $a + b$ is the same as $\frac{a+b}{1}$.

128. A **Mixed Expression** is an expression which has both integral and fractional literal parts; as, $a + \frac{b}{c}$ or $x + \frac{y+z}{y-z}$.

Rule. — A fraction may be changed to an integral or mixed expression by performing the indicated division.

Example 1. Reduce $\frac{6x^2 + 15x - 2}{3x}$ to a mixed expression.

Solution: Using the method of short division (§ 70),

$$\frac{6x^2 + 15x - 2}{3x} = \frac{6x^2}{3x} + \frac{15x}{3x} + \frac{-2}{3x} = 2x + 5 - \frac{2}{3x}. \quad \textit{Ans.}$$

Example 2. Reduce $\frac{12x^3 - 8x^2 + 4x - 5}{4x^2 + 3}$ to a mixed expression

Solution: Using the method of long division (§ 71),

$$\begin{array}{r|l}
 & 3x - 2 \\
\hline
4x^2 + 3 & 12x^3 - 8x^2 + 4x - 5 \\
 & 12x^3 \qquad\quad + 9x \\
\cline{2-2}
 & \qquad - 8x^2 - 5x - 5 \\
 & \qquad - 8x^2 \qquad\quad - 6 \\
\cline{2-2}
 & \qquad\qquad\quad - 5x + 1
\end{array}$$

$$\therefore \frac{12x^3 - 8x^2 + 4x - 5}{4x^2 + 3} = 3x - 2 + \frac{-5x + 1}{4x^2 + 3}.$$

$$= 3x - 2 - \frac{5x - 1}{4x^2 + 3}.$$

The first term of the numerator of the fraction in the result is negative. Change the signs in the numerator by multiplying the numerator by − 1. and also change the sign of the fraction. See § 126.

EXERCISE 79

Reduce the following to mixed expressions:

1. $\frac{315}{25}$. 2. $\frac{97}{14}$. 3. $\frac{263}{19}$. 4. $\frac{4350}{216}$.

5. $\dfrac{12a^2-16a+7}{4a}$, 7. $\dfrac{15p^2+12p-4}{3p}$.

6. $\dfrac{15m^3-6m^2+3m-8}{3m}$. 8. $\dfrac{30x^6+5x^4-15x^2-7}{5x^2}$.

9. $\dfrac{9y^2+5}{3y-1}$. 11. $\dfrac{x^3+2y^3}{x-y}$. 13. $\dfrac{x^4+y^4}{x-y}$.

10. $\dfrac{8p^2+3}{2p+3}$. 12. $\dfrac{a^3+8b^3}{a-2b}$. 14. $\dfrac{x^3+x^2y+6y^3}{x-2y}$.

15. $\dfrac{12a^2+5a+5}{4a-1}$. 16. $\dfrac{3a^3+8a^2-7}{a^2-2a-3}$.

TO REDUCE FRACTIONS TO THEIR LOWEST COMMON DENOMINATOR

129. In arithmetic, fractions having a common denominator may be added or subtracted without difficulty. Thus:

(*a*) $\frac{7}{15}+\frac{4}{15}=\frac{11}{15}$ (*b*) $\frac{9}{16}-\frac{3}{16}=\frac{6}{16}$.

Fractions which do not have the same denominator must be changed to equal fractions having a common denominator

(*a*) $\frac{9}{16}+\frac{3}{8}=\frac{9}{16}+\frac{6}{16}=\frac{15}{16}$, for $\frac{3}{8}=\frac{6}{16}$.

In algebra, also, fractions which do not have a common denominator must be changed to a common denominator before they can be added or subtracted.

130. To change fractions to a common denominator, a principle, easily illustrated arithmetically, is used.

If both terms of the fraction $\frac{6}{2}$ are multiplied by 4, the result is $\frac{24}{8}$. $\frac{6}{2}=\frac{24}{8}$ since both equal 3.

The value of the fraction is not changed; its *form only* is changed.

Rule. — **If the numerator and denominator of a fraction are both multiplied by the same number, the value of the fraction is not changed.**

131. Two fractions which have the same *value* but different *form* are called **Equivalent Fractions.**

Thus, $\frac{a}{b}$ and $\frac{ma}{mb}$ are equivalent fractions; they differ in form, but have the same value, since the second is obtained by multiplying both terms of the first by m.

132. The **Lowest Common Denominator** (**L. C. D.**) of two or more fractions is the lowest common multiple of their denominators

EXAMPLE 1. Reduce to their lowest common denominator

$$\frac{3\,mx}{2\,ab^2} \text{ and } \frac{5\,ny}{3\,a^3b}.$$

SOLUTION: 1. The L. C. M. of $2\,ab^2$ and $3\,a^3b$ is $6\,a^3b^2$. (§ 121)

2. To change the denominator $2\,ab^2$ into $6\,a^3b^2$, we must multiply $2\,ab^2$ by $3\,a^2$, that is by $6\,a^3b^2 \div 2\,ab^2$. In order not to change the value of the fraction, the numerator $3\,mx$ must also be multiplied by $3\,a^2$.

Then, $$\frac{3\,mx}{2\,ab^2} = \frac{3\,a^2 \cdot 3\,mx^2}{3\,a^2 \cdot 2\,ab^2} = \frac{9\,a^2mx}{6\,a^3b^2}. \qquad (\S\ 130)$$

3. For the second fraction, $6\,a^3b^2 \div 3\,a^3b = 2\,b$. Multiply both numerator and denominator by $2\,b$.

Then, $$\frac{5\,ny}{3\,a^3b} = \frac{2\,b \cdot 5\,ny}{2\,b \cdot 3\,a^3b} = \frac{10\,bny}{6\,a^3b^2}. \qquad (\S\ 130)$$

Rule. — **To reduce fractions to their lowest common denominator:**

1. Find the prime factors of the denominators.

2. Find the L. C. M. of the given denominators; this is the L. C. D.

3. For each fraction, divide the L. C. D. by the given denominator; multiply both numerator and denominator by the quotient.

EXAMPLE 2. Reduce to their lowest common denominator:

$$\frac{4\,x}{a^2 - 4} \text{ and } \frac{3\,a}{a^2 - 5\,a + 6}.$$

SOLUTION: 1. $\dfrac{4a}{a^2-4}=\dfrac{4a}{(a-2)(a+2)}.$

2. $\dfrac{3a}{a^2-5a+6}=\dfrac{3a}{(a-2)(a-3)}.$

3. The L. C. D. is $(a+2)(a-2)(a-3)$.

4. L. C. D. $\div (a-2)(a+2)=a-3$.

$$\therefore \frac{4a}{(a-2)(a+2)}=\frac{4a(a-3)}{(a-2)(a+2)(a-3)}.$$

5. L. C. D. $\div (a-2)(a-3)=(a+2)$.

$$\therefore \frac{3a}{(a-2)(a-3)}=\frac{3a(a+2)}{(a-2)(a+2)(a-3)}.$$

CHECK: The final fractions in steps 4 and 5 may be changed into the original fractions by cancellation.

EXERCISE 80

Reduce the following to equivalent fractions having their lowest common denominator:

1. $\frac{2}{3}$; $\frac{3}{4}$; $\frac{5}{6}$.
2. $\frac{3}{5}$; $\frac{7}{15}$; $\frac{9}{20}$.
3. $\dfrac{3a}{4}$; $\dfrac{5b}{6}$.
4. $\dfrac{4x}{3}$; $\dfrac{2y}{5}$; $\dfrac{3z}{2}$.
5. $\dfrac{5mn}{6}$; $\dfrac{7mp}{4}$; $\dfrac{3np}{5}$.
6. $\dfrac{2}{a}$; $\dfrac{3}{b}$; $\dfrac{5}{c}$.
7. $\dfrac{2x}{a}$; $\dfrac{3y}{a^2}$; $\dfrac{5z}{a^3}$.
8. $\dfrac{ab}{xy}$; $\dfrac{c}{xz}$; $\dfrac{d}{yz}$.
9. $\dfrac{1}{2m^3n}$; $\dfrac{2}{3mn^3}$; $\dfrac{6}{5m^2n^2}$.
10. $\dfrac{r^2s}{2m^2n}$; $\dfrac{r^3}{3mn^2}$; $\dfrac{s^3}{5m^4n^3}$.
11. $\dfrac{2a-5c}{9ab}$; $\dfrac{4a+3b}{12ac}$.
12. $\dfrac{3x-4z}{4xz^3}$; $\dfrac{6x-5z}{3x^2z}$.
13. $\dfrac{2ab}{a^2-b^2}$; $\dfrac{b}{a+b}$.
14. $\dfrac{4a^2}{4a^2-9}$; $\dfrac{2}{6a^2-9a}$.
15. $\dfrac{2x}{x+2y}$; $\dfrac{3y}{x-2y}$.
16. $\dfrac{3x}{5x-10}$; $\dfrac{5}{2x-4}$.

17. $\frac{x}{x^2-6x+8};\ \frac{x^2}{x^2-16}.$

19. $\frac{a+b}{a^2-2ab};\ \frac{a-b}{ab+b^2}.$

18. $\frac{3n}{n^3-8};\ \frac{5}{n^2-4n+4}.$

20. $\frac{3}{a+1};\ \frac{6}{a-1};\ \frac{9}{a^2-1}.$

21. $\frac{a+5}{a^2-a-6};\ \frac{a+3}{a^2+7a+10}.$

22. $\frac{a+3b}{a^2-7ab+12b^2};\ \frac{a-3b}{a^2-ab-12b^2}.$

23 $\frac{a-b}{x^2-2xy+y^2};\ \frac{x+y}{(a+b)(x-y)}.$

24 $\frac{x+y}{x^3-27y^3};\ \frac{xy}{x^2-9y^2}.$

25. $\frac{a+2}{a^2+2a-3};\ \frac{a-3}{a^2-3a+2};\ \frac{a+1}{a^2+a-6}.$

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ADDITION AND SUBTRACTION OF FRACTIONS

133. Example 1. Perform the indicated addition:

$$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}.$$

Solution: 1. The fractions cannot be added because they do not have a common denominator. By the methods of § 132, change the two fractions to equivalent fractions having their lowest common denominator.

2. The L. C. D. $= 12a^2b^3$. Multiply the terms of the first fraction by $12a^2b^3 \div 4a^2b$, or $3b^2$; and the terms of the second fraction by $12a^2b^3 \div 6ab^3$, or $2a$. Then,

3. $$\frac{4a+3}{4a^2b}+\frac{1-6b^2}{6ab^3}=\frac{3b^2\times(4a+3)}{3b^2\times 4a^2b}+\frac{2a\times(1-6b^2)}{2a\times(6ab^3)}$$
$$=\frac{(12ab^2+9b^2)}{12a^2b^3}+\frac{(2a-12ab^2)}{12a^2b^3}$$
$$=\frac{(12ab^2+9b^2)+(2a-12ab^2)}{12a^2b^3}$$
$$=\frac{12ab^2+9b^2+2a-12ab^2}{12a^2b^3}$$
$$=\frac{9b^2+2a}{12a^2b^3}.$$

CHECK: Let $a = 1$; $b = 1$.

$$\frac{4a+3}{4a^2b}=\frac{7}{4},\quad \frac{1-6b^2}{6ab^3}=\frac{-5}{6},\text{ and }\frac{7}{4}+\frac{(-5)}{6}=\frac{11}{12};$$

also,
$$\frac{9b^2+2a}{12a^2b^3}=\frac{9+2}{12}=\frac{11}{12}.$$

Rule. — To add or subtract fractions:

1. Reduce them, if necessary, to equivalent fractions having their lowest common denominator.

2. For the numerator of the result, combine the numerators of the resulting fractions, in parentheses, preceding each by the sign of its fraction.

3. For the denominator of the result, write the L. C. D.

4. Simplify the numerator by removing parentheses and combining like terms.

5. Reduce the result to lowest terms.

EXAMPLE 2. Simplify $\dfrac{5x-4y}{6}-\dfrac{7x-2y}{14}$.

SOLUTION:

1. $$\frac{5x-4y}{6}-\frac{7x-2y}{14}=\frac{7(5x-4y)}{42}-\frac{3(7x-2y)}{42}$$

2. $$=\frac{7(5x-4y)-3(7x-2y)}{42}$$

3. $$=\frac{35x-28y-21x+6y}{42}$$

4. $$=\frac{14x-22y}{42}$$

5. $$=\frac{2(7x-11y)}{42}=\frac{7x-11y}{21}.$$

CHECK. Let $x = 1$; $y = 1$.

$$\frac{5x-4y}{6}=\frac{5-4}{6}=\frac{1}{6},\quad \frac{7x-2y}{14}=\frac{5}{14},\text{ and }\frac{1}{6}-\frac{5}{14}=\frac{-8}{42}=\frac{-4}{21};$$

also,
$$\frac{7x-11y}{21}=\frac{7-11}{21}=\frac{-4}{21}.$$
The solution is correct.

NOTE. In the first step of the solution, the numerator and denominator of the first fraction are multiplied by 7, and of the second fraction, by 3.

EXERCISE 81

Perform the indicated additions and subtractions:

1. $\frac{2}{9}-\frac{5}{27}+\frac{1}{3}.$

2. $\frac{4}{15}+\frac{6}{5}-\frac{3}{20}.$

3. $\frac{5}{16}+\frac{3}{8}-\frac{1}{4}.$

4. $\frac{3}{14}+\frac{2}{21}-\frac{5}{84}.$

5. $\frac{5a}{4}+\frac{3a}{2}.$

6. $\frac{6m}{7}-\frac{3m}{14}.$

7. $\frac{15x}{8}-\frac{7x}{4}+\frac{3x}{2}.$

8. $\frac{a}{3}-\frac{b}{6}+\frac{c}{15}.$

9. $\frac{10r}{9}-\frac{2s}{3}+\frac{7t}{27}.$

10. $\frac{ab}{5}+\frac{bc}{10}-\frac{ac}{15}.$

11. $\frac{6a-5}{8}+\frac{3a+7}{12}.$

12. $\frac{2a-8}{7}+\frac{3a+5}{14}.$

13. $\frac{3m+4}{12}-\frac{2m+5}{9}.$

14. $\frac{4a-9}{9}-\frac{3a-8}{12}.$

15. $\frac{5a+1}{6}+\frac{2b+3}{8}-\frac{7c-4}{12}.$

16. $\frac{3a+4}{3}-\frac{4a-3}{4}+\frac{5a+2}{6}.$

17. $\frac{2a-3b}{9}-\frac{3a+b}{18}+\frac{4a-5b}{27}.$

18. $\frac{6m+1}{3}-\frac{5m-2}{6}+\frac{8m-3}{9}-\frac{7m+4}{12}.$

19. $\frac{4r-3s}{5}+\frac{6r+5s}{10}-\frac{5r+2s}{15}-\frac{3r-10s}{20}.$

20. $\frac{3t-2x}{4}-\frac{7t-8x}{6}+\frac{9t+4x}{8}-\frac{10t+7x}{9}.$

21. $\frac{3}{a}-\frac{5}{2a}$.

22. $\frac{2}{3x}+\frac{3}{4y}$.

23. $\frac{2}{a^2}-\frac{1}{ab}+\frac{3}{b^2}$.

24. $\frac{1}{r}+\frac{2}{s}-\frac{3}{t}$.

25. $\frac{a}{y}+\frac{b}{x}-\frac{c}{z}$.

26. $\frac{a}{2m^2}-\frac{b}{mn}-\frac{c}{5n^2}$.

27. $\frac{5a+1}{6a}+\frac{3a-2}{2a}$.

28. $\frac{10x+3y}{2x^2y}-\frac{3x+5y}{xy^2}$.

29. $\frac{x-y}{xy}+\frac{y-2z}{2yz}+\frac{z-3x}{3zx}$.

30. $\frac{2a-b}{ab}+\frac{2b-c}{bc}+\frac{2c-a}{ca}$.

31. $\frac{1}{x^2+x}-\frac{1}{x^2-x}$.

Solution: 1. $\frac{1}{x^2+x}-\frac{1}{x^2-x}=\frac{1}{x(x+1)}-\frac{1}{x(x-1)}$

2. $=\frac{(x-1)}{x(x+1)(x-1)}-\frac{(x+1)}{x(x+1)(x-1)}$

3. $=\frac{(x-1)-(x+1)}{x(x+1)(x-1)}$

4. $=\frac{x-1-x-1}{x(x+1)(x-1)}$

5. $=\frac{-2}{x(x^2-1)}=\frac{+2}{x(1-x^2)}$.

Notice that in line 2, the L.C.D. is $x(x+1)(x-1)$; that the numerator and denominator of the first fraction are multiplied by $(x-1)$, and of the second fraction by $(x+1)$; that parentheses are used in lines 2 and 3; that in line 5, the signs of both numerator and denominator are changed. Check by substitution.

32. Simplify $\frac{a+b}{4a^2-9b^2}-\frac{a-b}{(2a+3b)^2}$.

Solution: 1. $\frac{a+b}{4a^2-9b^2}-\frac{a-b}{(2a+3b)^2}$

2. $=\frac{a+b}{(2a-3b)(2a+3b)}-\frac{a-b}{(2a+3b)(2a+3b)}$

3. $=\frac{(a+b)(2a+3b)}{(2a-3b)(2a+3b)(2a+3b)}-\frac{(a-b)(2a-3b)}{(2a-3b)(2a+3b)(2a+3b)}$

4. $= \dfrac{(2a^2+5ab+3b^2)-(2a^2-5ab+3b^2)}{(2a-3b)(2a+3b)(2a+3b)}$

5. $= \dfrac{2a^2+5ab+3b^2-2a^2+5ab-3b^2}{(2a-3b)(2a+3b)(2a+3b)}$

6. $= \dfrac{10ab}{(2a-3b)(2a+3b)(2a+3b)}$.

CHECK: Let $a = 1$, $b = 1$.

$\dfrac{a+b}{4a^2-9b^2} = \dfrac{2}{-5} = -\dfrac{2}{5}$, $\dfrac{a-b}{(2a+3b)^2} = \dfrac{0}{25} = 0$, and $-\dfrac{2}{5} - 0 = -\dfrac{2}{5}$;

also, $\dfrac{10ab}{(2a-3b)(2a+3b)^2} = \dfrac{10}{(2-3)(5)^2} = \dfrac{10}{-25} = -\dfrac{10}{25} = -\dfrac{2}{5}$.

Notice that the indicated products in step 3 in the numerator are found and inclosed in parentheses in step 4. All of this solution should be done mentally.

33. $\dfrac{m}{m-1} - \dfrac{1}{m+1}$.

34. $\dfrac{r}{r+3} + \dfrac{r}{r-3}$.

35. $\dfrac{3}{5m-2} - \dfrac{2}{2m+3}$.

36. $\dfrac{3b}{3a-4} - \dfrac{5b}{5a+6}$.

37. $\dfrac{3y}{3y-x} - \dfrac{2x}{2x-3y}$.

38. $\dfrac{x}{2x+2y} + \dfrac{y}{3x-3y}$.

39. $\dfrac{1}{3a-7} - \dfrac{2}{6a+15}$.

40. $\dfrac{3}{4p-6} - \dfrac{4}{15p-12}$.

41. $\dfrac{2-x}{2+x} - \dfrac{2+x}{2-x}$.

42. $\dfrac{4p^2+1}{4p^2-1} - \dfrac{2p-1}{2p+1}$.

43. $\dfrac{1}{2a+b} - \dfrac{(2a-b)^2}{8a^3+b^3}$.

44. $\dfrac{x+3}{x-3} - \dfrac{x^3+27}{x^3-27}$.

45. $\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2} - \dfrac{16}{x^2-4}$.

46. $\dfrac{1}{a+b} + \dfrac{1}{a-b} - \dfrac{2a}{a^2-b^2}$.

47. $\dfrac{1}{a-x} - \dfrac{3x}{a^2-x^2} + \dfrac{ax}{a^3-x^3}$.

48. $\dfrac{m+n}{m-n} - \dfrac{m-n}{m+n} - \dfrac{4mn}{m^2-n^2}$.

49. $\dfrac{2}{x^2-5x+6} - \dfrac{5}{x^2+2x-15}$.

50. $\dfrac{a+1}{a^2-a-6}-\dfrac{a-4}{a^2-4a+3}+\dfrac{a+3}{a^2+a-2}.$

51. $\dfrac{3a+2}{6a^2-a-1}+\dfrac{a+3}{3a^2+7a+2}-\dfrac{a-2}{2a^2+3a-2}.$

52. $\dfrac{r}{r^2-6ar+9a^2}-\dfrac{a}{r^2+4ar-21a^2}.$

53. $\dfrac{x+2}{x^2+4x+3}-\dfrac{2(x-1)}{x^2+x-6}+\dfrac{x-3}{x^2-x-2}.$

54. $\dfrac{1}{a+1}-\dfrac{a}{a^2-a+1}+\dfrac{a^2-4}{a^3+1}.$

55. Simplify $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}.$

Solution: 1. Notice that b^2-a^2 is not in the same order as $a-b$. Change the signs of the denominator and also the sign of the fraction (§ 126)

2. $\dfrac{3}{a-b}+\dfrac{2b+a}{b^2-a^2}=\dfrac{3}{a-b}-\dfrac{a+2b}{a^2-b^2}.$

3. $=\dfrac{3}{a-b}-\dfrac{a+2b}{(a-b)(a+b)}$

4. $=\dfrac{3(a+b)-(a+2b)}{(a-b)(a+b)}$

5. $=\dfrac{3a+3b-a-2b}{(a-b)(a+b)}$

6. $=\dfrac{2a+b}{a^2-b^2}.$

Check by substitution.

56. $\dfrac{3a}{a^2-9}+\dfrac{5}{3-a}.$

57. $\dfrac{1}{y^2-xy}-\dfrac{1}{x^2-xy}.$

58. $\dfrac{x+4}{3x-6}-\dfrac{2x-5}{8-4x}.$

59. $\dfrac{a}{4m-m^2}+\dfrac{a}{m^2-16}.$

60. $\dfrac{3}{a+1}+\dfrac{3}{1-a}-\dfrac{6}{a^2-1}.$

61. $\dfrac{r}{r+2}-\dfrac{r}{2-r}-\dfrac{r^2}{r^2-4}.$

62. $\dfrac{a}{a+b}-\dfrac{b}{a-b}-\dfrac{2\,b^2}{b^2-a^2}$.

63. $\dfrac{2\,x}{6-x-x^2}+\dfrac{3\,x}{x^2+3\,x-10}$.

64. $13+\dfrac{6}{5}-\dfrac{1}{3}$.

65. $a-4+\dfrac{2+11\,a}{3\,a}$.

66. $\dfrac{x-y}{x+y}+1$.

67. $1-\dfrac{5\,a-b}{5\,a+b}$.

68. $3\,a+1-\dfrac{3-7\,a}{2\,a-3}$.

69. $a-b-\dfrac{a^3-b^3}{a^2+ab+b^2}$.

70. $3\,m+4-\dfrac{9\,m^2+16}{3\,m-4}$.

71. $r^2-rs+s^2-\dfrac{2\,s^3}{r+s}$.

MULTIPLICATION OF FRACTIONS

134. In arithmetic, the product of two fractions is the product of their numerators divided by the product of their denominators Thus, $\frac{3}{5}\times\frac{2}{11}=\frac{6}{55}$.

In algebra, the same rule is followed.

EXAMPLE 1. Multiply $\dfrac{5\,a}{2\,x}$ by $\dfrac{4\,x^2}{15\,a^2}$.

SOLUTION: 1. $\dfrac{5\,a}{2\,x}\cdot\dfrac{4\,x^2}{15\,a^2}=\dfrac{20\,ax^2}{30\,a^2x}$.

2. Reduce to lowest terms:

$$\frac{20\,ax^2}{30\,a^2x}=\frac{\overset{2\ x}{\cancel{20\,ax^2}}}{\underset{3\ a}{\cancel{30\,a^2x}}}=\frac{2\,x}{3\,a}.$$

It is customary to cancel the common factors in step 1 as in the following example.

EXAMPLE 2. Simplify $\dfrac{a^2+2\,a-3}{a^2-16}\cdot\dfrac{a^2-4\,a}{a^2-1}$.

SOLUTION: 1. $\dfrac{a^2+2\,a-3}{a^2-16}\cdot\dfrac{a^2-4\,a}{a^2-1}$

$$=\frac{(a+3)\overset{1}{\cancel{(a-1)}}}{\underset{1}{\cancel{(a-4)}}(a+4)}\cdot\frac{a\cancel{(a-4)}}{(a+1)\underset{1}{\cancel{(a-1)}}}$$

$$=\frac{a(a+3)}{(a+4)(a+1)}=\frac{a^2+3a}{a^2+5\,a+4}.$$

CHECK: Let $a = 2$.

$$\left.\begin{array}{l}\dfrac{a^2+2a-3}{a^2-16}=\dfrac{4+4-3}{4-16}=\dfrac{5}{-12}=\dfrac{-5}{12},\\[2ex] \quad\dfrac{a^2-4a}{a^2-1}=\dfrac{4-8}{4-1}=\dfrac{-4}{3},\end{array}\right\}\text{ and }\begin{array}{l}\dfrac{-5}{12}\cdot\dfrac{-4}{3}=\dfrac{+20}{36}\\[2ex] \qquad\qquad\;\;=\dfrac{5}{9};\end{array}$$

also, $$\frac{a^2+3a}{a^2+5a+4}=\frac{4+6}{4+10+4}=\frac{10}{18}=\frac{5}{9}.$$

Notice that the factors $(a-1)$ of the first numerator and of the second denominator are each divided by $(a-1)$, or, are cancelled; similarly the factors $(a-4)$ of the first denominator and the second numerator.

Rule. — **To find the product of two or more fractions:**

1. Find all of the prime factors of the numerators and denominators.

2. Divide out (cancel) factors common to a numerator and a denominator.

3. Multiply the remaining factors of the numerators for the numerator of the product, and of the denominators for the denominator of the product.

EXERCISE 82

Find the following indicated products:

1. $\dfrac{3}{14}\cdot\dfrac{21}{20}\cdot\dfrac{10}{9}.$

2. $\dfrac{5}{18}\cdot\dfrac{6}{10}\cdot\dfrac{12}{7}.$

3. $\dfrac{6a^2b}{15x^3y}\cdot\dfrac{5xy^3}{2ab}.$

4. $\dfrac{3am^2}{20b^2}\cdot\dfrac{25b^3}{3a^2m^2}.$

5. $\dfrac{5a}{4b}\cdot\dfrac{6b}{10c}\cdot\dfrac{8c}{12a}.$

6. $\dfrac{5a^2b}{2ac^2}\cdot\dfrac{b^2c}{3ba^2}\cdot\dfrac{6c^2a}{5b^2c}.$

7. $\dfrac{4a^6}{9b^3}\cdot\dfrac{15b^4}{7c^4}\cdot\dfrac{21c^5}{10a^5}.$

8. $\dfrac{27m^3y^2}{20m^4x}\cdot\dfrac{30n}{14x^2y^3}\cdot\dfrac{7x^4y^2}{18n^3}.$

9. $\dfrac{x^2-a^2}{x^2+2ax+a^2}\cdot\dfrac{2x+2a}{3x}.$

10. $\dfrac{4m^2-1}{m^3-16m}\cdot\dfrac{m^2+4m}{2m+1}.$

11. $\dfrac{a^2+2a-35}{6a^3}\cdot\dfrac{2a(a-3)}{(a+7)(a-3)}.$

12. $\dfrac{p^2+4p-45}{4pr-20r}\cdot\dfrac{3pr-3r^2}{p^2-81}.$

13. $\dfrac{a^2+3a-18}{a^4-8a^3+12a^2}\cdot\dfrac{2a^3-4a^2}{a^2-36}.$

14. $\dfrac{x^4-x^2}{x^3+8}\cdot\dfrac{x^2-2x+4}{x^3+2x^2+x}.$

15. $\dfrac{xy^2-y^3}{x^3+x^2y}\cdot\dfrac{x^2-xy-2y^2}{x^2-2xy+y^2}.$

16. $\dfrac{2a^2+5a+2}{2a^2+a-6}\cdot\dfrac{2a^2-3a}{6a^3+3a^2}.$

17. $\dfrac{6r^2-rs-2s^2}{12r^2+5rs-2s^2}\cdot\dfrac{8r^2-6rs+s^2}{4r^2-s^2}.$

18. $\dfrac{5m+10}{8m-4}\cdot\dfrac{3m-9}{10m+5}\cdot\dfrac{8m^2-2}{3m^2-12}.$

19. $\dfrac{x^4-y^4}{2x^2+2y^2}\cdot\dfrac{x^2+xy-2y^2}{x^2+3xy+2y^2}\cdot\dfrac{5x+5y}{x^2-2xy+y^2}.$

20. $\dfrac{8x^3-27}{4x^2-9}\cdot\dfrac{4x^2+12x+9}{8x^2+12x+18}\cdot\dfrac{4x-6}{6x^2+5x-6}.$

21. $\dfrac{m^4-1}{16m^4-9n^2}\cdot\dfrac{4m^2-3n}{2m^2+2}\cdot\dfrac{4m^2+3n}{m-1}.$

22. $\dfrac{2a^2-a}{4a^2-1}\cdot\dfrac{4a^2+4a+1}{5a}\cdot\dfrac{10a^2-10a}{4a^2-2a-2}.$

23. $\dfrac{1-x^2}{2x-4}\cdot\dfrac{x^2-4}{x^2-x-2}\cdot\dfrac{3x-6}{x^2+x-2}.$

24. $\dfrac{a^3-x^3}{a^2+2ax+x^2}\cdot\dfrac{5x}{3x-3a}\cdot\dfrac{6a+6x}{5a^2+5ax+5x^2}.$

25. $\dfrac{m^3+n^3}{m^3-n^3}\cdot\dfrac{n-m}{n+m}\cdot\dfrac{[(m+n)^2-mn]}{[(m-n)^2+mn]}.$

DIVISION OF FRACTIONS

135. In arithmetic, to divide a fraction by another fraction, we multiply the dividend by the inverted divisor. Thus:

$$(a)\quad \frac{9}{10} \div \frac{3}{5} = \frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{10}}} \times \frac{\cancel{5}}{\cancel{3}} = \frac{3}{2}.$$

$$(b)\quad 3\tfrac{2}{3} \div 2\tfrac{1}{5} = \frac{11}{3} \div \frac{11}{5} = \frac{\cancel{11}}{3} \times \frac{5}{\cancel{11}} = \frac{5}{3}.$$

In algebra, it is advisable to factor the expressions first, and then use the same process as in arithmetic.

Example. Divide $\dfrac{x^2 - x}{x^3 + 1}$ by $\dfrac{x^2 - 2x + 1}{x^3 - x^2 + x}$.

Solution: 1. $\dfrac{x^2 - x}{x^3 + 1} \div \dfrac{x^2 - 2x + 1}{x^3 - x^2 + x}$

2. $= \dfrac{x(x-1)}{(x+1)(x^2 - x + 1)} \div \dfrac{(x-1)(x-1)}{x(x^2 - x + 1)}$

3. $= \dfrac{x(\overset{1}{\cancel{x-1}})}{(x+1)(\underset{1}{\cancel{x^2 - x + 1}})} \cdot \dfrac{x(\overset{1}{\cancel{x^2 - x + 1}})}{(\underset{1}{\cancel{x-1}})(x-1)}$

4. $= \dfrac{x^2}{(x+1)(x-1)} = \dfrac{x^2}{x^2 - 1}.$

Check: Let $x = 2$.

$$\frac{x^2 - x}{x^3 + 1} = \frac{2}{9},\quad \frac{x^2 - 2x + 1}{x^3 - x^2 + x} = \frac{1}{6},\ \text{and}\ \frac{2}{9} \div \frac{1}{6} = \frac{4}{3};$$

also,
$$\frac{x^2}{x^2 - 1} = \frac{4}{3}.$$

Rule. — To divide one fraction by another:

1. Factor the numerators and denominators of the fractions.
2. Invert the divisor fraction.
3. Multiply the dividend fraction by the inverted divisor.

EXERCISE 83

Perform the following indicated divisions:

1. $\frac{5}{18} \div \frac{10}{9}$.

2. $\frac{7a}{15b} \div \frac{5a}{6b}$.

3. $\frac{21c^2}{32d} \div \frac{3c}{8d^2}$.

4. $\frac{15x^2y}{16xy^2} \div \frac{3x^2}{4y^2}$.

5. $\frac{32m^3}{15n^2} \div \frac{8}{35n^5}$.

6 $\frac{24a^4b^2}{5x^4y^3} \div 6a^2b^2$.

7. $\frac{3a}{x^2-6x+8} \div \frac{2a}{x^2-x-12}$.

8. $\frac{4r^2-25v^2}{16r^2-9v^2} \div \frac{2rv-5v^2}{4r^2+3rv}$.

9. $\frac{e^3-e}{3e^3+3} \div \frac{(e-1)^2}{6e^3-6e^2+6e}$.

10. $\frac{c^4-d^4}{(c-d)^2} \div \frac{(c^2+d^2)}{(c+d)}$.

11. $\frac{r^3-s^3}{r^2+rs+s^2} \div \frac{(r-s)^2}{4r+4s}$.

12. $\frac{(t+2s)^2}{(t-s)} \div \frac{ts+2s^2}{t^2-ts}$.

13. $\frac{a^2m+10am+21m}{a^3-4a^2+3a} \div \frac{a^2m^3-9m^3}{a^3-a^2}$.

14. $\frac{w^3-8}{w^2+7w+10} \div \frac{w^2+2w+4}{w^2+2w}$.

15. $\frac{a^2-5ab-14b^2}{a^2+5ab-24b^2} \div \frac{a^2-3ab-28b^2}{a^2-8ab+15b^2}$.

16. $\frac{m^3-y^3}{m^2y^2-y^4} \div \frac{m^3+m^2y+my^2}{my^2+y^3}$.

Perform the indicated multiplications and divisions:

17. $\frac{x^2+7xy+10y^2}{x^2+6xy+5y^2} \cdot \frac{x+y}{x^2+4xy+4y^2} \div \frac{1}{x+2y}$.

18. $\frac{a^2-b^2}{a^2-3ab+2b^2} \cdot \frac{ab-2b^2}{b^2+ab} \div \frac{(a-b)^2}{a(a-b)}$.

19. $\frac{(2r-3s)^2}{r^2+4rs+4s^2} \cdot \frac{4r^2-4s^2}{4r^2-9s^2} \div \frac{3rs-3s^2}{5r^2+10rs}$.

20. $\frac{am^4-an^4}{bm^3-bn^3} \div \frac{m^2+n^2}{m^2-n^2} \cdot \frac{bm^2+bmn+bn^2}{am^2+2amn+an^2}$.

136. Sometimes there are mixed expressions in an algebra problem. These mixed expressions should always be reduced to fractional form as in arithmetic.

Example 1. Simplify $12\frac{3}{8} \div 3\frac{1}{4}$.

Solution:
$$12\tfrac{3}{8} \div 3\tfrac{1}{4} = \frac{99}{8} \div \frac{13}{4} = \frac{99}{\underset{2}{\cancel{8}}} \times \frac{\overset{1}{\cancel{4}}}{13} = \frac{99}{26}.$$

Example 2 Simplify $\left(5 - \frac{a^2 - 19x^2}{a^2 - 4x^2}\right) \div \left(3 - \frac{a - 5x}{a - 2x}\right)$.

Solution: In the first parentheses, the fraction is to be subtracted from 5, and in the second, the fraction is to be subtracted from 3. (Use rule, § 133.)

1. $$\left(5 - \frac{a^2 - 19x^2}{a^2 - 4x^2}\right) \div \left(3 - \frac{a - 5x}{a - 2x}\right)$$

2. $$= \left(\frac{5(a^2 - 4x^2) - (a^2 - 19x^2)}{a^2 - 4x^2}\right) \div \left(\frac{3(a - 2x) - (a - 5x)}{a - 2x}\right)$$

3. $$= \left(\frac{5a^2 - 20x^2 - a^2 + 19x^2}{a^2 - 4x^2}\right) \div \left(\frac{3a - 6x - a + 5x}{a - 2x}\right)$$

4. $$= \left(\frac{4a^2 - x^2}{a^2 - 4x^2}\right) \div \left(\frac{2a - x}{a - 2x}\right)$$

5. $$= \frac{\cancel{(2a - x)}(2a + x)}{\cancel{(a - 2x)}(a + 2x)} \times \frac{\cancel{(a - 2x)}}{\cancel{(2a - x)}} = \frac{2a + x}{a + 2x}.$$

Check: Let $a = 1$; $x = 1$.

$$\left.\begin{aligned} 5 - \frac{a^2 - 19x^2}{a^2 - 4x^2} &= 5 - \frac{1 - 19}{1 - 4} = 5 - \frac{-18}{-3} = 5 - 6 = -1, \\ 3 - \frac{a - 5x}{a - 2x} &= 3 - \frac{1 - 5}{1 - 2} = 3 - \frac{-4}{-1} = 3 - 4 = -1, \end{aligned}\right\} \begin{aligned} &\text{and} \\ &(-1) \div (-1) = 1; \end{aligned}$$

also, $$\frac{2a + x}{a + 2x} = \frac{2 + 1}{1 + 2} = \frac{3}{3} = 1.$$

Note. In such examples, first perform carefully the indicated additions and subtractions *within the parentheses* and, afterwards, the multiplications and divisions

EXERCISE 84

Perform the indicated operations:

1. $(3+\frac{2}{3})\cdot(3+\frac{3}{11})$.

2. $(5-\frac{1}{4})\cdot(2-\frac{2}{19})$.

3. $\left(1-\frac{x}{y}\right)\cdot\left(\frac{y}{y^2-x^2}\right)$.

4. $\left(a+\frac{b^2}{a}\right)\cdot\left(1-\frac{b^2}{a^2+b^2}\right)$.

5. $\left(\frac{2}{a}-\frac{3}{b}\right)\div\left(\frac{2}{a}+\frac{3}{b}\right)$.

6. $\left(\frac{a}{b}-\frac{c}{d}\right)\div\left(\frac{a}{b}+\frac{c}{d}\right)$.

7. $\left(2-\frac{a}{b}\right)\left(\frac{3b^2}{8b^3-a^3}\right)$.

8. $\left(\frac{a^2}{5b}-\frac{a}{2}\right)\div\left(\frac{2a^2}{3b}+\frac{a}{2}\right)$.

9. $\left(b-\frac{a^2}{b}\right)\div(b-a)$.

10. $\left(\frac{a}{4b}-\frac{b}{a}\right)\cdot\left(\frac{8a^2b^2}{a+2b}\right)$.

11. $\left(2-\frac{y}{x+y}\right)\div\left(2-\frac{x}{x+y}\right)$.

12. $\left(4-\frac{1}{x+1}\right)\div\left(16+\frac{7}{x^2-1}\right)$.

13. ✓ $\left(1-\frac{2ab}{a^2+ab+b^2}\right)\cdot\left(1-\frac{2b^3}{a^3+b^3}\right)$.

14. ✓ $\left(a+1-\frac{30}{a}\right)\div\left(a-4-\frac{5}{a}\right)$.

$(a-4)(a+1)$

15. ✓ $\left(r+s+\frac{s^2}{r}\right)\cdot\left(2+\frac{2s^3}{r^3-s^3}\right)$.

16. ✓ $\left(6+\frac{5x+2}{x^2-1}\right)\div\left(2+\frac{1}{x-1}\right)$.

17. ✓ $\left(\frac{2m^2+7m-15}{2m^2-3m-14}\right)\cdot\left(\frac{2m^2-19m+42}{8m-12}\right)\div\left(\frac{m^2-m-30}{3m+6}\right)$.

18. $\left(\frac{1}{v^2}-\frac{2}{vx}+\frac{1}{x^2}\right)\left(1+\frac{2v}{x-v}\right)\div\left(\frac{x}{v}-\frac{v}{x}\right)$.

19. $\left(1+\frac{5a+4}{a^2-3a-4}\right)\cdot\left(1+\frac{a-16}{a^2-a}\right)\div\left(\frac{a^2+6a+8}{a^2+a}\right)$.

20. $\left(\frac{x^3+27}{x^2+x-12}\right)\div\left\{\frac{x^2-3x+9}{x^2+2x-8}\cdot\frac{x^2+x-6}{3x-9}\right\}$.

COMPLEX FRACTIONS

137. A **Complex Fraction** is a fraction having one or more fractions in either or both of its terms; as,

$$(a)\ \frac{a}{b-\frac{c}{d}};\qquad (b)\ \frac{\left(\frac{a}{a-b}-\frac{a}{a+b}\right)}{\left(\frac{b}{a-b}+\frac{a}{a+b}\right)}.$$

138. A complex fraction is simply a case of division of fractions. The problems are very similar to those in the last paragraph. The numerator and denominator should be simplified separately and then the division performed.

Example 1. Simplify $\dfrac{a}{b-\frac{c}{d}}$.

Solution: This means divide a by the result obtained by subtracting $\frac{c}{d}$ from b.

$$\frac{a}{b-\frac{c}{d}}=\frac{a}{\left(\frac{bd-c}{d}\right)}=a\cdot\left(\frac{d}{bd-c}\right)=\frac{ad}{bd-c}.$$

Check: Let $a=1$, $b=2$, $c=1$, $d=1$.

$$\frac{a}{b-\frac{c}{d}}=\frac{1}{2-\frac{1}{1}}=\frac{1}{2-1}=\frac{1}{1}=1;$$

also, $$\frac{ad}{bd-c}=\frac{1}{2-1}=1.$$

Example 2. Simplify $\dfrac{\left(\frac{a}{a-b}-\frac{a}{a+b}\right)}{\left(\frac{b}{a-b}+\frac{a}{a+b}\right)}$.

This means: find the difference of the fractions in the numerator, and the sum of the fractions in the denominator; divide the first result by the second. The work may be arranged as follows:

Solution: 1. $$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}$$

2. $$=\frac{a(a+b)-a(a-b)}{(a-b)(a+b)}\div\frac{b(a+b)+a(a-b)}{(a-b)(a+b)}$$

3. $$=\frac{a^2+ab-a^2+ab}{(a-b)(a+b)}\div\frac{ab+b^2+a^2-ab}{(a-b)(a+b)}$$

4. $$=\frac{2\,ab}{(a-b)(a+b)}\div\frac{a^2+b^2}{(a-b)(a+b)}$$

5. $$=\frac{2\,ab}{\cancel{(a-b)(a+b)}}\times\frac{\cancel{(a-b)(a+b)}}{a^2+b^2}=\frac{2\,ab}{a^2+b^2}.$$

Check: Let $a=2$; $b=1$.

$$\frac{\dfrac{a}{a-b}-\dfrac{a}{a+b}}{\dfrac{b}{a-b}+\dfrac{a}{a+b}}=\frac{\dfrac{2}{1}-\dfrac{2}{3}}{\dfrac{1}{1}+\dfrac{2}{3}}=\frac{\dfrac{4}{3}}{\dfrac{5}{3}}=\frac{4}{3}\times\frac{3}{5}=\frac{4}{5};$$

also, $$\frac{2\,ab}{a^2+b^2}=\frac{4}{4+1}=\frac{4}{5}.$$

EXERCISE 85

1. $\dfrac{\frac{2}{3}-\frac{3}{4}}{\frac{2}{3}+\frac{3}{4}}.$

2. $\dfrac{1+\frac{1}{6}}{3-\frac{1}{12}}.$

3. $\dfrac{2-\frac{1}{2}}{1-\frac{1}{3}}.$

4. $\dfrac{4}{6-\frac{2}{3}}.$

5. $\dfrac{3-\frac{1}{2}}{1-\frac{1}{3}},$

6. $\dfrac{\dfrac{3\,ab}{x}}{\dfrac{6\,a^2b}{x^2}}.$

7. $\dfrac{\dfrac{x+y}{2}}{\dfrac{x^2-y^2}{6}}.$

8. $\dfrac{3}{1+\dfrac{2}{x}}.$

9. $\dfrac{\dfrac{2}{m}}{2-\dfrac{1}{m}}.$

10. $\dfrac{1+\dfrac{1}{2\,a}}{a-\dfrac{1}{4\,a}}.$

11. $\dfrac{p-\dfrac{1}{p^2}}{1-\dfrac{1}{p}}.$ ✓

12. $\dfrac{\dfrac{1}{x^3}-x}{\dfrac{1}{x^2}-1}.$

13. $$\frac{\frac{a}{b}-\frac{c}{d}}{\frac{a}{b}+\frac{c}{d}}.$$

15. $$\frac{x+1-\frac{20}{x}}{x-2-\frac{8}{x}}.$$

17. $$\frac{a-\frac{2b^2}{a-b}}{a+\frac{b^2}{a+2b}}.$$

14. $$\frac{\frac{2x}{3y}-2+\frac{3y}{2x}}{\frac{2}{y}-\frac{3}{x}}.$$

16. $$\frac{x^2-13+\frac{36}{x^2}}{x+1-\frac{6}{x}}.$$

18. $$\frac{\frac{x^3}{y^2}-\frac{8y}{x}}{\frac{x}{y}+2+\frac{4y}{x}}.$$

19. $$\frac{\frac{m}{n}+\frac{m+n}{m-n}}{\frac{n}{m}+\frac{m+n}{m-n}}.$$

22. $$\frac{1-\frac{8(x^2+y^2)}{9x^2-y^2}}{1-\frac{2(x+2y)}{3x+y}}.$$

here

20. $$\frac{a^2-a+1-\frac{2}{a+1}}{a+\frac{1}{a+1}}.$$

23. $$\frac{6+\frac{17x+22}{x^2-3x-4}}{2+\frac{9}{x-4}}.$$

21. $$\frac{\frac{a}{a+b}}{1+\frac{b}{a-b}}.$$

24. $$\frac{a-\frac{a-x}{1+ax}}{1+\frac{a^2-ax}{1+ax}}.$$

25. $$\frac{\frac{2a-b}{a+3b}+\frac{2a+b}{a-3b}}{\frac{2a-b}{a-3b}-\frac{2a+b}{a+3b}}.$$

XI. SIMPLE EQUATIONS — (*Continued*)

FRACTIONAL EQUATIONS

139 If the unknown number, or numbers, of an equation *do not appear* in the denominator of a fraction, the equation is called an **Integral Equation**; as, $3x - 5 = 2x + 7$.

140. If the unknown number, or numbers, of an equation *do appear* in the denominator of a fraction, the equation is called a **Fractional Equation**; as, $\frac{3}{x} - 5 = \frac{2}{x} + 7$.

141. A fractional equation may be changed into an integral equation by **Clearing of Fractions.**

EXAMPLE. Solve the equation $\frac{3x-1}{4} - \frac{4x-5}{5} = 4 + \frac{7x+5}{10}$.

SOLUTION: 1. The lowest common multiple of 4, 5, and 10 is 20. Multiply both members of the equation by 20.

2. $\overset{5}{\cancel{20}} \cdot \frac{(3x-1)}{\cancel{4}} - \overset{4}{\cancel{20}} \cdot \frac{(4x-5)}{\cancel{5}} = 20 \cdot 4 + \overset{2}{\cancel{20}} \cdot \frac{(7x+5)}{\cancel{10}}$.

3. $\therefore 5(3x-1) - 4(4x-5) = 80 + 2(7x+5)$.

4. $\therefore 15x \quad 5 - 16x + 20 = 80 + 14x + 10$.

5. $\therefore 15 - x = 90 + 14x$.

6. $\therefore -15x = 75$.

7. D_{-15}: $x = -5$.

CHECK: Does $\frac{3(-5)-1}{4} - \frac{4(-5)-5}{5} = 4 + \frac{7(-5)+5}{10}$?

Does $\frac{-16}{4} - \frac{-25}{5} = 4 + \frac{-30}{10}$? Does $-4 + 5 = 4 - 3$? Yes.

EXERCISE 86

Solve the following equations:

1. $\frac{x+5}{2}-\frac{x+1}{4}=3.$

2. $\frac{y-7}{5}+2=\frac{y+8}{10}.$

3. $\frac{11+m}{6}-\frac{10-m}{3}=1.$

4. $\frac{1}{3}(a+5)-4=\frac{a-10}{4}.$

5. $\frac{1}{3}(r+2)-\frac{1}{5}(r-2)=2.$

6. $\frac{c-2}{4}-\frac{c-4}{6}=\frac{2}{3}.$

7. $\frac{w+12}{9}=\frac{w-9}{2}.$

8. $\frac{m+11}{6}-\frac{10-m}{3}=-1.$

9. $\frac{t-1}{3}+3=\frac{t+14}{9}.$

10 10. $\frac{s+18}{4}-\frac{3}{7}(s-3)=4.$

11. $\frac{1}{10}(7w+1)-\frac{1}{4}(w-9)=1.$

12. $\frac{3(x-1)}{10}-\frac{5x+7}{3}=\frac{17}{6}.$

13. 3 $\frac{4b+1}{2}-\frac{1}{3}(2b+3)=\frac{5b-1}{4}.$

-1 14. $\frac{n+5}{2}-\frac{1}{3}(4n+9)=\frac{1}{9}(5n+8).$

15. $\frac{6t-1}{4}-\frac{1}{10}(8t+3)=\frac{1}{5}(4t-3).$

16. $\frac{7x-8}{14}-\frac{7x+6}{4x}=\frac{x-5}{2}-\frac{4x+9}{7x}.$

SOLUTION: 1. Multiply both members by $28x$, the L. C. M of the denominators

2. $\overset{2x}{\cancel{28x}}\cdot\frac{(7x-8)}{\cancel{14}}-\overset{7}{\cancel{28x}}\cdot\frac{(7x+6)}{\cancel{4x}}=\overset{14x}{\cancel{28x}}\cdot\frac{(x-5)}{\cancel{2}}-\overset{4}{\cancel{28x}}\cdot\frac{(4x+9)}{\cancel{7x}}$

3. $\therefore 2x(7x-8)-7(7x+6)=14x(x-5)-4(4x+9).$

4. $\therefore 14x^2-16x-49x-42=14x^2-70x-16x-36.$

5. $\therefore -65x-42=-86x-36.$

6. $\therefore +21x=+6.$

7. $\therefore x=\frac{6}{21}=\frac{2}{7}.$

Check the solution either by substitution or by going over the steps carefully.

It is essential that roots of fractional equations be checked. For reasons that are given in a later course in algebra, whenever the *apparent root* makes the denominator of any fraction zero, *that number is not a root at all.*

17. 3 $\frac{6}{a}-\frac{1}{a}=\frac{5}{3}.$

18. $\frac{5}{4m}-\frac{2}{3m}=\frac{7}{48}.$

19. $\frac{3}{4x}+\frac{7}{x}=15\frac{1}{2}.$

20. $\frac{x-2}{5x}=\frac{4}{x}-2.$

21. $\frac{12}{x}-\frac{2x+8}{5x}+3=0.$

1 **22.** $\frac{5t+4}{2t}-\frac{11t-2}{6t}=3.$

1 **23.** $\frac{3m-5}{4}-\frac{9m-7}{12}=\frac{2}{3m}.$

1 **24.** $\frac{30}{7}-\frac{10x+9}{21x}+\frac{7}{3x}=0.$

25. $\frac{5x-4}{5}-\frac{10x+9}{10}=\frac{51}{6x}.$

3 **26.** $\frac{r-3}{2r}=\frac{1}{3}-\frac{3r-7}{2r}.$

27 Solve the equation $\frac{2}{x-2}-\frac{5}{x+2}-\frac{2}{x^2-4}=0.$

SOLUTION: 1. The L. C. M. of $(x-2)$, $(x+2)$, and (x^2-4) is (x^2-4). Multiplying both members of the equation by x^2-4,

2. $\overset{(x+2)}{\cancel{(x^2-4)}}\cdot\frac{2}{\cancel{(x-2)}}-\overset{(x-2)}{\cancel{(x^2-4)}}\cdot\frac{5}{\cancel{(x+2)}}-\overset{1}{\cancel{(x^2-4)}}\cdot\frac{2}{\cancel{(x^2-4)}}=(x^2-4)\cdot 0.$

3. $\therefore 2(x+2)-5(x-2)-2=0.$

4. $\therefore 2x+4-5x+10-2=0.$

5. $\therefore -3x+12=0.$

6. $\therefore -3x=-12$, or, $x=4.$

$2\frac{1}{3}-2$

$\frac{4}{3}-\frac{6}{8}=$

28. $\frac{12}{x-2}=3.$

29. $\frac{10}{m-3}=\frac{9}{m-5}.$

30 $\frac{x+5}{x-3}+\frac{4}{x-3}=5.$

31. $\frac{5x+1}{2x-3}+3=\frac{x}{2x-3}.$

32. $\frac{4-x}{1-x}=\frac{12}{3-x}+1.$

33. $\frac{2x}{3x-4}=\frac{4x+5}{6x-1}-\frac{3}{3x-4}.$

34. $\frac{15x^2-5x-8}{3x^2+6x+4}=5.$

35. $\frac{6x+5}{2x^2-2x}+\frac{2}{x^2-1}=\frac{3x}{x^2-1}.$

36. $\dfrac{3m}{2m+3} - \dfrac{2m}{2m-3} = \dfrac{2m^2-15}{4m^2-9}.$

37. $\dfrac{9}{3t-5} - \dfrac{2}{t-2} = \dfrac{1}{t-3}.$

38. $\dfrac{3}{r-2} - \dfrac{4}{2r-1} = \dfrac{1}{r+4}.$

39. $\dfrac{2(y-7)}{y^2+3y-28} + \dfrac{y-2}{y-4} - \dfrac{y+3}{y+7} = 0.$

40. $\dfrac{a+2}{a-4} - \dfrac{2a-3}{a+3} = \dfrac{26-a^2}{a^2-a-12}.$

41. $\dfrac{3m}{2m-6} - \dfrac{4}{5m-15} = \dfrac{1}{2}.$

42. $\dfrac{3a}{2a-5} - \dfrac{7}{3a+1} = \dfrac{3}{2}.$

43. $\dfrac{2z+1}{2z-16} - \dfrac{2z-1}{2z+12} = \dfrac{9z+17}{z^2-2z-48}.$

44. $\dfrac{2t+7}{6t-4} - \dfrac{3t-5}{9t+6} = \dfrac{17t+7}{9t^2-4}.$

45. $\dfrac{3x-2}{x+3} = \dfrac{36-4x}{x^2-9} - \dfrac{2+3x}{3-x}.$

46. Solve the equation $\dfrac{6x+1}{15} - \dfrac{2x-4}{7x-16} = \dfrac{2x-1}{5}.$

SOLUTION: 1. Clear of fractions only partially at first by multiplying by the L. C. M. of 15 and 5.

M_{15}: $\quad 6x+1-\dfrac{15(2x-4)}{7x-16} = 6x-3.$

2. Transposing and uniting terms, $\quad 4 = \dfrac{30x-60}{7x-16}.$

3. $M_{(7x-16)}$: $\quad 28x-64 = 30x-60.$

4. Completing the solution, $\quad x = -2.$

Check it by substitution.

47. $\frac{2a-1}{2}-\frac{a+2}{2a+5}=\frac{6a-5}{6}$.

48. $\frac{y}{3}+\frac{2y^2}{3y-4}=\frac{9y-2}{9}$.

49. $\frac{2x+7}{14}-\frac{5x-4}{3x+1}=\frac{x+6}{7}$.

50. $\frac{5t+1}{5}-\frac{4t+7}{5t+8}-\frac{3t-2}{3}=0$.

EXERCISE 87

1. Divide 56 into two parts such that five eighths of the greater shall exceed seven twelfths of the less by 6.

2. If the base of a certain rectangle be increased by 2 feet, the altitude is equal to one third of the result. The perimeter of the rectangle is 36 feet. Find the base, altitude, and area of the rectangle.

3. A has $52 and B has $38 After giving B a certain sum, A has left only three sevenths as much money as B then has. What sum was given to B?

4. Divide 45 into two parts such that the sum of four ninths of the greater and two thirds of the smaller shall equal 24.

5. The denominator of a certain fraction exceeds the numerator by 27; if 9 be subtracted from both terms of the fraction, the value of the fraction becomes $\frac{1}{4}$. Find the fraction.

6. A's age is three eighths of B's, and 8 years ago it was two sevenths of B's age. Find their ages at present.

7. Washington was admitted to the Union 18 years before Oklahoma, and may therefore be said to be 18 years older than Oklahoma as a state. One fourth of Washington's age in 1911 exceeded Oklahoma's age by $1\frac{1}{2}$ years. Find the year when each was admitted.

35

8. If a certain number be diminished by 23, one fourth of the result is as much less than 37 as the number is greater than 56. Find the number.

9. If the number of states admitted to the Union since its formation by the 13 original states is diminished by 9, the quotient obtained by dividing that number by 2 equals the original number of states. Find the present number of states (1912).

10. The numerator of a certain fraction is 6 less than the denominator; if the denominator is increased by 1 and the resulting fraction be multiplied by 3, the product equals $\frac{5}{4}$. Find the fraction.

11. Find the angle such that 3 times its complement increased by two thirds of its supplement equals 137°.

12. I buy some bulbs from a seed store for $3, paying 75¢ per dozen for one variety, and 50¢ per dozen for another variety. The number of the first variety purchased exceeds the number of the second variety by 18. Find the number of each variety purchased.

13. If a railroad train consists of a certain type of passenger engine, one parlor car, and five sleeping cars, its value is $129,200. The value of each sleeping car exceeds the value of the engine by $300; the value of the parlor car is five sixths of the remainder when the cost of the engine is diminished by $100. Find the value of the engine, the parlor car, and of a sleeping car.

14. A man has $3000 invested, part at 5 % and part at 6 %. His total income per year is $157. How much has he invested at each rate?

15. In 1912, the "age" of Maine was $4\frac{2}{11}$ times that of Wyoming; in 1920, it will be $3\frac{1}{3}$ times it. Find when each state was admitted to the Union.

$270-3x+\frac{2}{3}(180-x)$

16. The denominator of a certain fraction is 7 less than the numerator; if 5 be added to the numerator, the value of the fraction becomes $\frac{9}{5}$. Find the fraction.

17. The income at 5% on one sum of money exceeds by $35.50 the income at 4% on a sum which is $350 less than the first. Find the two sums invested.

18. The denominator of a certain fraction exceeds the numerator by 5; if the denominator be decreased by 20, then the resulting fraction, increased by 1, is equal to twice the original fraction. Find the fraction.

19. The supplement of a certain angle divided by its complement gives as quotient $2\frac{1}{2}$. Find the angle.

20. If twice a certain number be diminished by 5 and the result be divided by the number, the quotient exceeds 1 by a fraction whose numerator is 7 more than the number and whose denominator is 3 less than the number. Find the number.

142. Work Problems. If a man can do a piece of work in 8 days, then in one day he can do *one eighth* of it, and in three days he can do *three eighths* of it.

If a man can do a piece of work in x days, then in one day he can do $\frac{1}{x}$ part of it, and in 5 days he can do $5\times\frac{1}{x}$ or $\frac{5}{x}$ part of it.

EXERCISE 88

1. If a man can plow a field in 15 days, what part can he plow in one day? in 4 days? in x days?

2. If a machine can do a piece of work in x days, how much can it do in one day? in 7 days?

3. A can do a piece of work in 5 days, and B the same work in 8 days.

(*a*) How much can A do in one day? in x days?
(*b*) How much can B do in one day? in x days?

(*c*) How much can they do together in one day? in x days?

(*d*) How much can A do in 2 days?

(*e*) How much can B do in 3 days?

(*f*) How much can they do together if A works 2 days and B 3 days?

Equations

4. A can do a piece of work in 10 hours and B can do it in 5 hours; how long will it take them to do it together?

SOLUTION: 1. Let $x =$ the number of hours it will take them to do it together.

2. A does $\frac{1}{10}$ in 1 hour; ∴ he will do $\frac{x}{10}$ in x hours.

B does $\frac{1}{5}$ in 1 hour; ∴ he will do $\frac{x}{5}$ in x hours.

∴ they will do $\left(\frac{x}{10}+\frac{x}{5}\right)$ in x hours.

3. They complete the task in x hours. Represent the whole task by $\frac{10}{10}$ of itself or by 1.

4. $$\therefore \frac{x}{10}+\frac{x}{5}=1.$$

5. M_{10} $$x+2x=10.$$

6. $$3x=10,\ x=3\tfrac{1}{3} \text{ hours, or 3 hours and 20 minutes.}$$

5. A painter can paint a house alone in 5 days, and an apprentice can do it alone in 15 days. In how many days can they do it if they work together?

6. A man can plow a certain field in 6 days; his son can plow it in 9 days. How long will it take them to plow the field if they work together?

7. In a newspaper office there is one machine which can print the morning issue of the paper in 2 hours, and another which can do it in 3 hours. In how many hours can they turn out the edition, if they are run together?

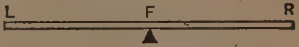
8. A can do a piece of work in 15 hours, and B can do it in 18 hours. If A works for 7 hours, how many hours must B work to complete the task?

9. A can do a piece of work in 15 hours, while B can do it in 25 hours. After A has worked a certain time, B completes the work. If B works 9 hours longer than A, how long did A work?

10. A can do a piece of work in 18 days. If he and B can do three fifths of it in 6 days, how long will it take B alone to do the work?

143. Problems about the Lever. A teeter board is one form of *lever*. The point which supports the lever is called the *fulcrum;* the parts of the lever to the right and left of the fulcrum are called the *lever arms.*

If the weights L and R just balance, it is well known that, if R moves farther to the right, while L is stationary, then the right side goes down; and if R moves toward the fulcrum, then the right side goes up. Thus, the influence of R upon the lever depends both upon the weight of R and its distance from the fulcrum.

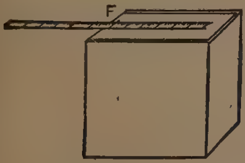


The influence of a weight upon a lever is called its *leverage.* It can be shown that the *leverage of a weight is measured by the product of the weight and its distance from the fulcrum.*

Thus, if R weighs 50 pounds and is 4 feet from the fulcrum, its leverage is 200; and, if L weighs 80 pounds and is $2\frac{1}{2}$ feet from the fulcrum, its leverage is $80 \times 2\frac{1}{2}$ or 200 also. The two will balance.

This truth may be tested in the following manner.

1. Remove a side and an end from a crayon box. Balance a stiff ruler on the edge of the box (the fulcrum).



2. Place two pennies 6 inches from the fulcrum on the left side. Find where four pennies must be placed on the right side of the fulcrum to balance them. (It should be 3 inches to the right.) Notice that $3 \times 4 = 12$ and that $6 \times 2 = 12$; that the leverages are equal.

3. Find where 3 pennies must be placed, on the right, to balance the 2 pennies on the left. Find the leverage of the 3 pennies and compare it with the leverages in step 2.

prod of weight & distance

dis ÷ rate = time   dis ÷ time = rate

rate × time = dis

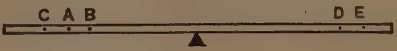
4. Keeping the two pennies 6 inches to the left, place two others 2 inches to the right, and 2 four inches to the right. The ruler should be in perfect balance again.

The leverage of the first two pennies on the right is 2×2 or 4; of the second two, is 2×4 or 8; $4 + 8 = 12$.

Example. Suppose that the weights and distances in the figure are:

A. 40 pounds; distance 5 feet.
B. 45 pounds; distance 4 feet.
C. 55 pounds; distance 6 feet.
D. 60 pounds; distance 6 feet.
E. 50 pounds; distance x feet.

Where must *E* be placed so that the lever will balance?



Solution: 1. The leverages are:

A. $5 \times 40 = 200.$ *C* $6 \times 55 = 330.$ *E.* $50 \times x = 50x.$
B. $4 \times 45 = 180.$ *D.* $6 \times 60 = 360.$

$$\begin{array}{llllllllll} & & A & & B & & C & & D & & E \\ 2. & \therefore & 200 & + & 180 & + & 330 & = & 360 & + & 50x. \end{array}$$

$$\therefore 350 = 50x$$

$$\therefore \quad 7 = x.$$

Rule. — To make a simple lever balance:

The sum of the leverages of all weights (forces) on one side of the fulcrum must equal the sum of the leverages of all weights (forces on the other side of the fulcrum.

EXERCISE 89

1. A boy weighing 70 pounds sits 6 feet from the fulcrum and balances a boy who is sitting $3\frac{1}{2}$ feet from the fulcrum on the other side. Find the weight of the second boy.

2. A, weighing 96 pounds, sits $5\frac{1}{2}$ feet to the left of the fulcrum. If B weighs 66 pounds, where must he sit on the right in order to balance A?

3. A, who weighs 92 pounds, and B, who weighs 115 pounds, wish to sit at the ends of a teeter board which is 9 feet long. How far from A must the fulcrum be placed so that they will balance?

(4 − x) 30

4. A boy wishes to carry two heavy packages over his shoulder by balancing them at the ends of a stiff rod which is 4 feet long If one package weighs 20 pounds and the other 30 pounds, how far from the end upon which the 20 pound package is carried must the rod rest upon his shoulder?

5. Three children, weighing 62, 75, and 89 pounds respectively, arrange themselves upon a teeter board, the first sitting 4 feet from the fulcrum, and the second 5 feet from the fulcrum on the same side. Where must the third sit in order to balance the other two?

6. Three boys, A weighing 73 pounds, B weighing 95 pounds, and C weighing 65 pounds, sit on a teeter. A is 5 feet from the fulcrum on the left side, B is on the other side 4 feet away, C is on the left side 4 feet away. Can two other boys weighing 80 pounds and 115 pounds respectively arrange themselves one on each side and at equal distances from the fulcrum so as to balance the teeter? Where must the boy weighing 80 pounds sit?

144. Additional Distance, Rate, and Time Problems.

EXERCISE 90

1. What is meant by "the rate"? "the time"? "the distance"? (§ 83.)

2. Give a simple arithmetical problem involving time, rate, and distance.

3. What is the rule for finding the
   (*a*) distance when the rate and time are known?
   (*b*) rate when the time and distance are known?
   (*c*) time when the rate and distance are known?

x = rate of [illegible]
x + 5 .. fast train – $\frac{100}{x}$ = slow

$\frac{100}{x+5}$ = fast

4. The rate of one train exceeds that of another by 5 miles per hour. Let r represent the rate of the slow train.

(*a*) Express the rate of the faster train.

(*b*) Express the time required by each train in going 100 miles.

3/5 x 5. The rate of one train is $\frac{3}{5}$ that of another. Let x represent the rate of the faster train.

(*a*) Express the time each requires for a trip of 50 miles.

(*b*) Form an equation to express the fact that the time of the slow train exceeds that of the faster train by 1 hour.

6. The time required by one train in going a certain distance is $\frac{3}{2}$ that of another train. Express the time of the slow train by t.

t = slow t = [illegible]

(*a*) Express the rate at which each train travels in going a distance of 100 miles in the time mentioned.

(*b*) Form an equation to express the fact that the rate of the faster train exceeds the rate of the slow train by 20 miles per hour.

Equations

3x + 8 = [illegible]

7. The rate of an express train is three times that of a slow train. It covers 180 miles in 8 hours less time than the slow train. Find the rate of each train.

8. A messenger starts out to deliver a package to a point 24 miles distant, at the rate of 8 miles per hour. At what rate must a second messenger travel to arrive at the same time as the first messenger, if he starts 1 hour after him?

9. The rate of a passenger train exceeds twice the rate of a freight train by 5 miles per hour. It can go 350 miles while the freight train goes 150 miles. Find the rate of each train.

10. A man just missed a train. He knew that it would stop at a station 15 miles distant from the central station and

decided to try to catch it by going to the second station in an automobile. If the train runs at the rate of 20 miles per hour, at what rate must he travel in the automobile in order to arrive at the station 10 minutes ahead of the train, if it takes him 5 minutes to get the automobile and if it is 20 miles to the station by road?

River Problems

On a river, the direction in which the water is flowing is called *downstream*, and the opposite direction is called *upstream*. When going downstream, a boat is carried along by the *current* of the river and whatever force is exerted within the boat; when coming upstream, its progress is retarded by the current of the river. This is something like the effect of the wind upon a person who is walking; when going with the wind, he is carried along by it, and when going against the wind, he is retarded by it.

11. The rate of the current of a river is 3 miles per hour: (*a*) at what rate will some boys go downstream if their own rowing is at the rate of 5 miles per hour in still water? (*b*) upstream?

12. How long will it take the boys in Example 11: (*a*) to go 24 miles downstream? (*b*) 24 miles upstream? (*c*) for the trip down and back?

13. (*a*) How long will it take the same boys to go d miles downstream? (*b*) d miles upstream? (*c*) Form an equation to express the fact that the time down and back is 5 hours. (*d*) Find d from the equation in step (*c*).

14. Some boys who can row 4 miles an hour in still water made a trip on a river whose current is 2 miles an hour. If it took them 8 hours for the trip, how far did they go?

15. A party take a trip in a motor boat which runs at the rate of 15 miles an hour. They take 3 hours for the trip. What distance did they go, if the rate of the current is 3 miles an hour?

EXERCISE 91

Supplementary Problems

1. Three times the difference between one fourth and one tenth of a certain number exceeds five by one fifth of the number. Find the number.

2. A's age 11 years from now divided by his age 11 years ago is the fraction $\frac{16}{5}$. Find his present age.

3. The width of a room is three fifths of its length; if 12 feet be added to the width and taken from the length, the room will be a square. Find its dimensions.

4. Five lines radiate from a point making angles such that the second is one half of the first, the third is twice the first, the fourth is the sum of the second and third, and the fifth is three times the third. Find the angles. (See § 13.)

5. Find the three angles of a triangle if the second angle is one half of the remainder obtained by diminishing the first angle by 1°, and if the third angle is $\frac{4}{3}$ of the remainder obtained by diminishing the first angle by 7°. (See § 13.)

6. Two men, A and B, 57 miles apart, travel towards each other, B starting 20 minutes after A. A travels at the rate of 6 miles an hour and B at the rate of 5 miles an hour. How far will each have traveled when they meet?

7. Washington's Monument, the highest piece of masonry in the world, consists of a main shaft surmounted by a pyramid. The height of the main shaft exceeds that of the pyramid by 445 feet; if the height of the pyramid be decreased by 5 feet and the result be divided by the height of the main shaft, the quotient is the fraction $\frac{1}{10}$. Find the height of the two parts of the monument.

8. The rate of an express train is five thirds of that of a slow train. It travels 36 miles in 32 minutes less time than the slow train. Find the rate of each train.

9. If the third of three consecutive even numbers is divided by each of the first two in turn, the difference of the fractions obtained is equal to the quotient of 7 divided by three times the first number. Find the numbers.

10. A workman does one third of a piece of work in 5 days; he and a second workman complete the task together in 4 days. How manys days would it take the second man to do the work alone?

11. Some boys row on a river whose current is known to be $2\frac{1}{2}$ miles per hour. They find that it takes them as long to go upstream 2 miles as downstream 7 miles. What is their rate of rowing?

3/5(6 − 2) . 4/7 (2x + 9

12. A's age exceeds twice B's age by 7 years. The sum of three fifths of B's age 2 years ago and of four sevenths of A's age 4 years from now is 26 years. Find their present ages.

13. The income on one sum of money at $4\frac{1}{2}\%$ and the income on a sum \$600 greater at $3\frac{1}{2}\%$ together amount to \$421 per year. Find the total amount invested.

HINT: $4\frac{1}{2}\% = \frac{9}{2}\% = \frac{9}{200}$.

14. The rate of an express train is three halves that of a slow train. It covers 270 miles in 3 hours less time than the slow train. Find the rate of the train.

15. A water reservoir can be filled by an old pump in 12 hours. After a new one is installed, it is found that the reservoir can be filled by the two pumps together in 4 hours. How long would it take the new pump alone to fill the reservoir? [illegible]

16. A man invests a sum of money in $4\frac{1}{2}\%$ stock and a sum \$180 greater than the first in $3\frac{1}{2}\%$ stock. If the incomes from the two investments are equal, find the sums invested.

17. Some boys are rowing on a river whose current is 5 miles per hour in one stretch and 3 miles in another. They find when going downstream that they can go 4 miles where the current is rapid in the same time that they can go 3 miles where the current is slower. Find the rate at which they row in still water?

The following three equations arise in the solution of three problems in applied mathematics. Solve them.

✓ 18. $80T - \frac{1}{2}gT^2 = 80(T-2) - \frac{1}{2}g(T-2)^2$, where $g = 32$ Determine T.

19. $\frac{5 \times 77}{x} = 13.6 + \frac{5-x}{x}$. Get the result to two decimals.

✓ 20. $\frac{9v}{10} + \frac{.0013\,\overset{x}{x}\,v}{10} = xv$. Determine x.

SOLUTION OF LITERAL EQUATIONS

145 A **Literal Equation** is one in which some or all of the known numbers are represented by letters.

Example 1. $2x + a = 7a - x$.

The problem is to determine a number x, such that the equation is satisfied for all values of a.

Solution: 1. $2x + a = 7a - x$.
2. Transposing, $3x = 6a$.
3. D_3: $x = 2a$.
Check: Does $2(2a) + a = 7a - 2a$?
Does $4a + a = 7a - 2a$? Yes.

This solution means that the number x is always double the number a.

Example 2. Determine the number x in the equation

$$(b - cx)^2 - (a - cx)^2 = b(b - a).$$

Solution: 1. $(b - cx)^2 - (a - cx)^2 = b(b - a)$.
2. Expanding, $(b^2 - 2bcx + c^2x^2) - (a^2 - 2acx + c^2x^2) = b^2 - ab$.
3. $\therefore b^2 - 2bcx + c^2x^2 - a^2 + 2acx - c^2x^2 = b^2 - ab$.
4. Transposing and uniting terms, $2acx - 2bcx = a^2 - ab$.
5. Factoring, $2cx(a - b) = a(a - b)$.
6. $D_{2c(a-b)}$: $\frac{2cx(a-b)}{2c(a-b)} = \frac{a(a-b)}{2c(a-b)}$.
7. $\therefore x = \frac{a}{2c}$.

$x^2 -$

$x - a$

CHECK: Does $\left(b-\frac{ac}{2c}\right)^2-\left(a-\frac{ac}{2c}\right)^2=b(b-a)?$

Does $b^2-ab+\frac{a^2}{4}-a^2+a^2-\frac{a^2}{4}=b^2-ab?$ Yes.

NOTE. After simplifying the equation until it takes the form of the equation in step 5, divide both members of the resulting equation by the coefficient of x.

EXERCISE 92

Solve the following equations for x:

1. $3x-5a=2(a+x)$.
2. $5(x-3b)=3x-11b$.
3. $a(3bx-2a)=b(2a-3bx)$.
4. $5rx-6s=3(rx-s)$.
5. $ax-ac=bx-bc$.
6. $ax-a^2=-b(b+x)$.
7. $r(x-r)=s(s+2r-x)$.
8. $c(x-c^2)=d(d^2-x)$.
9. $x-1-\frac{x-2}{m}=\frac{1}{m^2}$.
10. $\frac{m^2}{nx}-\frac{1}{m}=\frac{1}{n}-\frac{n^2}{mx}$.
11. $\frac{a-x}{5bx}=\frac{10b-a}{15ax}-\frac{1}{3a}$.
12. $\frac{x^2-b}{ax}-\frac{b-x}{a}=\frac{2x}{a}-\frac{b}{x}$.
13. $2n\left(\frac{n}{2}+\frac{mx}{n}\right)=4\left(m^2+\frac{n^2}{4}\right)$.
14. $\frac{x-a}{x}+\frac{2x}{x-a}=3$.
15. $\frac{3x-4}{3x+4}=\frac{5m-2n}{5m+2n}$.
16. $\frac{2x+3a}{2x-3b}=\frac{3x+4b}{3x-4a}$.
17. $\frac{x+n}{x-m}-\frac{x-n}{x+m}=\frac{2(m+n)^2}{x^2-m^2}$.
18. $\frac{2nx-3}{nx-1}=5-\frac{9nx+2}{3nx-1}$.
19. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a^2-b^2}{b(x-b)}$.
20. $\frac{ax-b}{bx}+\frac{bx+a}{ax}=2+\frac{a-b}{abx}$.
21. $\frac{10x-3a}{3a}=\frac{9x-5a^3}{3a^3}-\frac{3(x+2a^3)}{a^3}$.
22. $\frac{x}{2}-\frac{a-2bcx}{4bc}=\frac{5x}{6c}-\frac{8ac-8bx-9a}{12bc}$.

23. $\frac{x-b}{x-2a}-\frac{x+b}{x+2a}=\frac{4a^2}{x^2-4a^2}.$

24. $\frac{3x(a-b)}{x^2-b^2}-\frac{a-2b}{x+b}=\frac{a-b}{x-b}.$

25. $\frac{2x}{3x-a}-\frac{3a}{x+2a}=\frac{2x^2+ax}{3x^2+5ax-2a^2}.$

146. Applied Algebra. Some of the equation problems given up to this point illustrate one use of algebra; an unknown number may sometimes be found by means of an equation if numerical relations involving the number are known. In order to solve the equations that arise, skill in the fundamental operations (§ 16), factoring, and fractions, is necessary.

In § 17 and § 94, another application of algebra is illustrated; a *formula* is often used to express in convenient form a rule of computation. In *deriving formulæ* and in using them, all of the algebra so far studied, and much more, frequently, is necessary.

147. Deriving and Using Formulæ. Three examples of the methods of deriving formulæ will be given, besides those in § 17.

EXAMPLE 1. Derive a formula for the total area of the walls of a room in terms of the height, length, and width.

SOLUTION: 1. Let h = the height of the room in feet.
Let w = the width of the room in feet.
Let l = the length of the room in feet.
Let S = the total area of the walls in square feet.
2. Then, hw = the area of one end wall in square feet.
hw = the area of the other end wall.
hl = the area of one side wall.
hl = the area of the other side wall.
3. $\therefore S = hw + hw + hl + hl = 2hw + 2hl.$
$\therefore S = 2h(w + l).$

Thus, if the height is 9 feet, the width 14 feet, and the length 18 feet, the total area of the walls is

$$S = 2 \times 9 \times (14 + 18) = 18(32) = 576 \text{ square feet.}$$

Example 2. Derive formulæ for two numbers whose sum is a and such that the larger exceeds the smaller by b.

Solution: Let s = the smaller number.

Then, $s + b$ = the larger number.

2. $$\therefore s + (s + b) = a, \text{ or } 2s + b = a.$$

$$\therefore 2s = a - b, \text{ or } s = \frac{a - b}{2}.$$

$$\therefore s + b = \frac{a - b}{2} + b = \frac{a + b}{2}.$$

The smaller number is $\frac{a - b}{2}$ and the larger number is $\frac{a + b}{2}$.

Thus, if a is 25 and b is 7, the smaller number is $\frac{25 - 7}{2}$ or $\frac{18}{2}$, or 9, and the larger is $\frac{25 + 7}{2}$, or 16.

$$9 + 16 = 25 \text{ and } 16 = 9 + 7.$$

Example 3. Derive a formula for the *rate* in terms of the *amount*, the *principal*, and the *time*.

Solution: 1. The formula for the amount (§ 44) is

$$A = P + \frac{Prt}{100}.$$

2. M_{100}: $$100\,A = 100\,P + Prt.$$

3. S_{100P}: $$100\,A - 100\,P = Prt = (Pt)r.$$

4. D_{Pt}: $$\frac{100\,A - 100\,P}{Pt} = r, \text{ or } r = \frac{100(A - P)}{Pt}.$$

This formula enables us to find r when A, P, and t are known. Thus, if a man receives \$3500 at the end of 6 years from an investment of \$2400, what rate of simple interest has his money earned?

Here, $A = \$3500$, $P = \$2400$, and $t = 6$.

$$\therefore r = \frac{\cancel{100}(3500 - 2400)}{\underset{24}{\cancel{2400}} \times 6} = \frac{1100}{144} = 7.6+;$$

that is, the rate is 7.6 %.

In the following list of examples, a number of formulæ, taken from physics, chemistry, geometry, and engineering, are given. It is impossible to show in this text how these formulæ are derived, as that calls for special knowledge of these various subjects.

Thurs & Friday

EXERCISE 93

(*a*) Express each of the first 6 formulæ in words.

(*b*) Solve each of them for the letters indicated.

1. $A = ab$. (*a*) Solve for *a*; (*b*) for *b*. (§ 17)

2. $A = \frac{ab}{2}$. (*a*) Solve for *a*; (*b*) for *b*. (§ 17)

3. $C = 2\pi r$. Solve for *r*. (§ 17)

4. $V = lwh$. (*a*) Solve for *w*; (*b*) for *h*. (§ 17)

5. ✓ $V = \frac{1}{3}bh$. (*a*) Solve for *b*; (*b*) for *h*. (§ 17)

6. ✓ $F = \frac{9}{5}C + 32$. Solve for *C*. (§ 84)

7. ✓ $A = \frac{a(b+c)}{2}$ is a formula from geometry.

   (*a*) Find *A* when $a = 15$; $b = 24$; and $c = 20$.
   (*b*) Find *c* when $A = 550$; $b = 30$; and $a = 22$.
   (*c*) Solve the formula for *a*; (*d*) for *b*.

8. $A = P + \frac{Prt}{100}$.

   (*a*) Solve for *P*, (*b*) for *t*.

9. $T = \frac{1}{a} + t$ is a formula from physics.

   (*a*) Solve it for *t*. (*b*) for *a*.

10. ✓ $mg - T = mf$ is a formula from physics.

    (*a*) Solve it for *T*; (*b*) for *f*; (*c*) for *m*.

11. $u = \left(\frac{m}{M+m}\right)v$ is a formula from physics.

    (*a*) Solve it for *m*; (*b*) for *M*.

12. ✓ $C = \frac{Kab}{b-a}$ is a formula from physics.

    (*a*) Solve it for *a*; (*b*) for *b*.

just

work no time [illegible]

13. $s = \dfrac{c-b}{b-a}$ is a formula from physics.

(*a*) Solve it for b; (*b*) for a.

14. $h = k(1 + \frac{1}{273}t)$ is a formula from physics.

(*a*) Solve it for t; (*b*) for k.

15. $l = \dfrac{L-M}{Mt}$ is a formula from physics.

(*a*) Solve it for L; (*b*) for M.

16. $w = \dfrac{5a}{1 + \dfrac{l^2}{600d^2}}$ is a formula from engineering.

(*a*) Find w, correct to two decimals, when $a = 18$, $l = 12$, and $d = 10$.

(*b*) Solve the formula for a.

17. $p = \dfrac{ad^2}{t} + d$ is a formula from engineering.

(*a*) Find p when $a = .56$, $d = \frac{1}{4}$, $t = \frac{3}{8}$.

(*b*) Solve the formula for t.

18. $C = \dfrac{E}{R + \dfrac{r}{n}}$ is a formula from physics.

(*a*) Simplify it in the right member.

(*b*) Solve it for n; (*c*) for r.

19. $F = \dfrac{mv^2}{gr}$ is a formula from physics.

(*a*) Find F when $m = 150$, $v = 25$, $g = 32$, and $r = 5$.

(*b*) Solve it for r; (*c*) for m.

20. $\dfrac{1}{f} = \dfrac{1}{p} + \dfrac{1}{q}$ is a formula from physics.

(*a*) Find p when $f = 30$ and $q = 40$.

(*b*) Solve it for p; (*c*) for q.

XII. GRAPHICAL REPRESENTATION

148. Certain mathematical relations may be represented by means of geometrical figures. Paper ruled as in the figures below is used; such paper is called **Coördinate Paper**

Figure 1 is an illustration of this manner of comparing numbers. The lines represent the lengths of some of the rivers of America. The side of one square represents 200 miles; from this fact, the lengths of the rivers may be determined

Such representations of number relations are called **Graphical Representations,** or simply **Graphs.**

EXERCISE 94

1. Determine the lengths of the rivers in Fig. **1.**

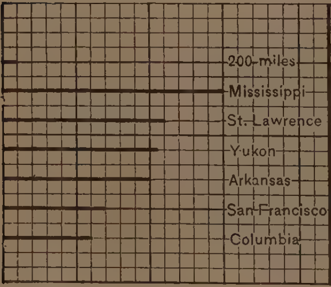


Fig. 1

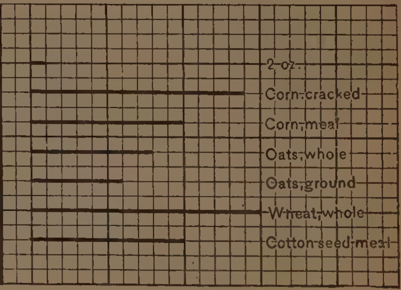
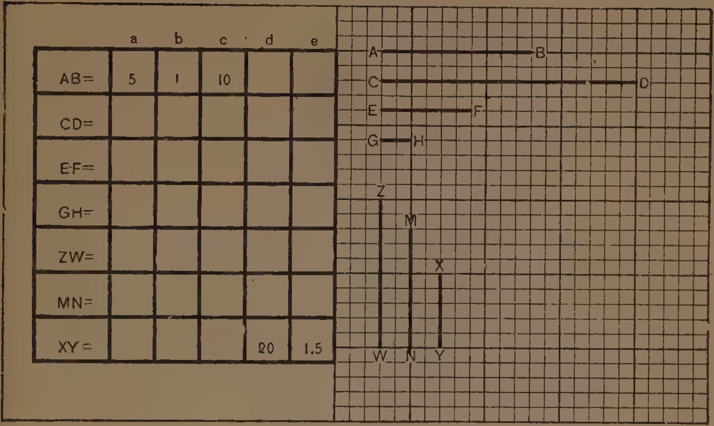


Fig. 2

2. In Fig. 2, the weight per quart of certain grains is represented Find the weight of each kind of grain.

3. Find the length of each line in Fig. 3, and arrange your results in the form of a table; when:



	a	b	c	d	e
AB=	5	1	10		
CD=					
EF=					
GH=					
ZW=					
MN=					
XY=				20	1.5

200 FIG. 3 200

Represent by means of lines the following sets of statistics:

4. A family used the following amounts of gas during the months of the year:

January	3600 cu. ft.	July	2300 cu. ft
February	3200 cu. ft.	August	2200 cu. ft
March	2900 cu. ft.	September	3100 cu. ft.
April	3100 cu. ft.	October	3200 cu. ft.
May	2900 cu. ft.	November	2800 cu. ft.
June	2900 cu. ft.	December	2200 cu. ft.

HINT. Let the side of one square represent 100 cu. ft., or 200 cu. ft., then represent the various given numbers by lines of appropriate length.

5. The same family used the following amounts of electricity during the months of the year.

January	27 k.w.	July	10 k.w.
February	27 k.w.	August	8 k.w.
March	24 k.w.	September	18 k.w
April	19 k.w.	October	42 k.w
May	15 k.w.	November	49 k.w
June	9 k.w.	December	45 k.w.

A k.w. is a "kilowatt-hour," the unit used for measuring the quantity of electricity consumed.

6. The following statistics give in cents the cost per pound of certain common articles of food in 1910.

Wheat	2.5	Egg	20
Rye	3	Bacon	22.5
Navy Beans	6	Milk	4
Potatoes	1.3	Apples	5
Rice	10	Oatmeal	10
Beefsteak	17	Buckwheat	2.5

7. The average hourly velocity of wind in miles at the following points in the United States is:

Boise, Idaho	4	Portland, Me.	5
Atlanta, Ga.	9	Vicksburg, Miss.	6
Buffalo, N. Y.	11	Philadelphia, Pa.	10
Duluth, Minn.	7	Chicago, Ill.	9
Omaha, Neb.	8	Boston, Mass.	11

8. The average annual precipitation (rain and melted snow), in inches, at the following cities is:

San Diego, Cal.	10	Mobile, Ala.	62
Denver, Colo.	14	Flagstaff, Ariz.	23
Springfield, Ill.	37	Salt Lake City, Utah	16
Dubuque, Ia.	34	Wilmington, N.C.	51
Baltimore, Md.	43	Portland, Me.	42.5

149. In drawing the graph of a set of statistics, the paper is usually prepared as in Fig. 4. Two lines are drawn at right angles. The numbers to be represented are examined in order to decide what number shall be represented by the side of one square. The number represented by one side of a square is called the **Unit**.

Instead of drawing full lines to represent the various numbers, it is customary to place points where the lines of proper length would end. Finally these points are connected by a smooth line. One line, that at 11 A.M., is drawn in full in this graph.

EXAMPLE 1. Draw a graph representing the temperature readings at the hours indicated.

6 A.M.	− 8°	11 A.M.	+ 7°	4 P.M.	+ 9°
7 A.M.	− 6°	12 M.	+ 8°	5 P.M.	+ 8°
8 A.M.	− 2°	1 P.M.	+ 9°	6 P.M.	+ 6°
9 A.M.	+ 2°	2 P.M.	+ 10°	7 P.M.	+ 4°
10 A.M.	+ 5°	3 P.M.	+ 10°	8 P.M.	0

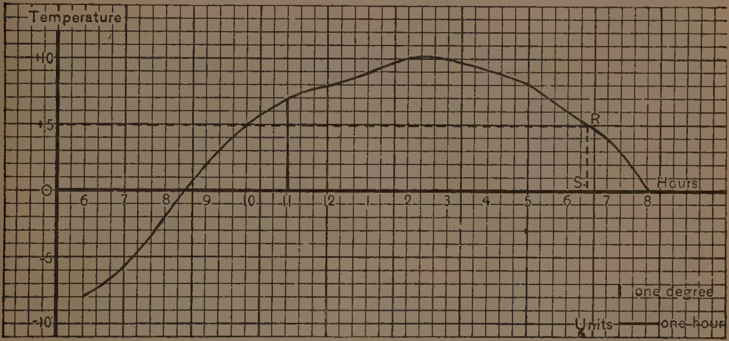


FIG. 4

Notice that the units of time and of temperature are indicated; that the line upon which the hours are indicated is placed near the center of the paper to allow for the *negative* temperatures, that the vertical line is placed at the left edge to allow space for all of the hours of the day; that the line connecting the points is made a smooth curve. Review § 24

150. The graph in Fig. 4 may be used to illustrate another advantage of graphical representation.

A graph may assist in finding new information.

What was the approximate temperature at 6.30 P.M.?

SOLUTION: 1. On the graph, the point *S* would indicate the time 6.30

2. The point *R*, above it on the graph, indicates the temperature.

3. *R* is opposite + 5° on the vertical line.

4. The approximate temperature then at 6.30 P.M. was 5°.

EXERCISE 95

1. Draw a graph representing the following temperature readings:

6 A.M.	+ 12°	1 P.M.	+ 16°
7 A.M.	+ 14°	2 P.M.	+ 12°
8 A.M.	+ 18°	3 P.M.	+ 6°
9 A.M.	+ 20°	4 P.M.	0°
10 A.M.	+ 22°	5 P.M.	− 4°
11 A.M.	+ 22°	6 P.M.	− 6°
12 M.	+ 19°	7 P.M.	− 7°

2. From the graph in 1, find the approximate temperature at 7.30, 8.30, 11.30, 2.30, 5.30.

3. Get the temperature readings in your own school district to-morrow, and draw the graph.

4. In order to protect his orchard from a frost, a man tried to warm the air in the orchard by burning oil heaters. Below are given the temperature readings within the heated area and outside of it. Plot the two sets of readings on the same sheet, making one line for the readings within the heated area, and one for the other readings.

Time	Temperature		Time	Temperature	
	Inside	Outside		Inside	Outside
8.00	35°	35°	10.30	36°	32°
8.15	38°	35°	11 00	35°	31°
8.30	39°	33°	11.30	36.5°	30°
8.45	40°	32°	12.00	36.5°	28°
9.00	40°	34°	12.30	34.5°	27°
9.30	40°	34°	1.00	34.5°	27°
10.00	40°	34°			

5. Savings banks and trust companies pay compound interest on money left with them; that is, interest upon interest.

The following table shows the amount of money which must be saved annually at 4 % compound interest to amount to $1000 in a definite number of years. (To simplify the problem even dollars are taken.)

10 years	$80	25 years	$23
15 years	$48	30 years	$17
20 years	$32	35 years	$13

6. Assume that a train leaves station A at 1 P.M. and that it arrives at F and J, the distances of which from A are given, at the times indicated. Draw the graph representing the time when the train reaches points between these three stations, making the graph a **broken** line, (/‾‾‾) From the graph, determine when the train reaches the other points, thus making a time-table for the train between A and H.

A.	0 miles.	1 P.M.	F.	60 miles.	3 P.M.
B.	6 miles.		G.	80 miles.	
C.	15 miles.		H.	104 miles.	
D.	24 miles.		I.	120 miles.	
E.	42 miles.		J.	140 miles.	5 P.M.

HINT. If possible, lay off the distances on the horizontal line, making the unit 2 or 6 miles, and the time on the vertical line, making the unit 6 or 12 minutes.)

7. Complete the following table showing the number of dollars simple interest at 6 % on $100 for the number of years indicated and draw the graph.

1 year; interest $6.
2 years; interest $12.
3 years; interest ? 18
4 years; interest ? 24
5 years; interest ? 30
6 years; interest ? 36
7 years; interest ? 42
8 years; interest ? 48

8. From the graph, determine the interest for 5 years and 6 months.

151. Certain terms used in mathematics in connection with graphical representation will now be given. Referring to Fig. 5 below: the lines XX' and YY', drawn at right angles, are called **Axes**: XX', the **Horizontal Axis**, and YY', the **Vertical Axis**; the point O is called the **Origin**. From P, perpendiculars are drawn to the axes; the line PR is called the **Ordinate** of P, and the line PS is called the **Abscissa of** P; together they are called the **Coördinates** of P The axes are called **Coördinate Axes.** On the axes, the scale used is indicated. The distances

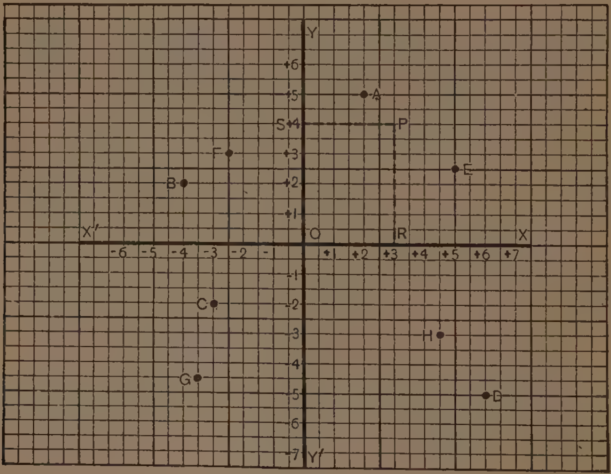


FIG. 5

on OX are positive, on OX' are negative; on OY are positive, and on OY' are negative. The abscissa of P is 3 and the ordinate is 4: the point P is called **The Point (3, 4).** Notice that the *abscissa* is written *first*, and the *ordinate, second.*

EXERCISE 96

1. What is the abscissa of the point A? B? C? D?

2. What is the ordinate of the point A? B? C? D?

3. What are the coördinates of E? F? G? H?

4. Prepare a piece of graph paper similar to that used for Fig 5. Locate the points: (3, 6); (4, 2); (5, −3); (−2, 5); (−4, −6).

5. Locate the points $A:(1, 1)$; $B:(2, 6)$; $C:(11, 6)$; $D:(10, 1)$.

Draw the lines AB, BC, CD, and AD. What figure is formed? Draw the diagonals of the figure, and find the coördinates of the point where they meet.

6. Locate the points $A:(3, -5)$; $B:(4, -2)$; $C:(7, -2)$; $D:(8, -5)$; $E:(7, -8)$; $F:(4, -8)$. Draw the lines joining the points in order. What figure is formed?

152. The numbers which are represented graphically in mathematics are usually connected by an equation.

Example. Draw the graph of the equation

$$\frac{x}{2}+\frac{y}{4}=1.$$

Solution. 1. Solve the equation for y.

M_4: $2x+y=4.$

S_{2x}: $y=4-2x.$

2. Select any value of x and find the corresponding value of y. Thus, if $x=1$, $y=4-2=2$. Similarly:

when	$x=$	-5	-3	-1	0	$+2$	$+4$	$+6$	$+7$
then	$y=$	$+14$	$+10$	$+6$	$+4$	0	-4	-8	-10

3. Use the pairs of numbers so obtained as coördinates of points; thus, locate the points $(-5, +14)$; $(-3, +10)$; $(-1, +6)$; etc.

4. Draw the line connecting the points. (Graph on page 214.)

Notice that the graph seems to be a straight line. The coördinates of the point A on the graph are $+3.5$ and -3. Do these satisfy the equation?

$$\frac{x}{2}+\frac{y}{4}=1.$$

$$\text{Does } \frac{3.5}{2}+\frac{-3}{4}=1?$$

$$\text{Does } 1.75+(-.75)=1? \quad \text{Yes.}$$

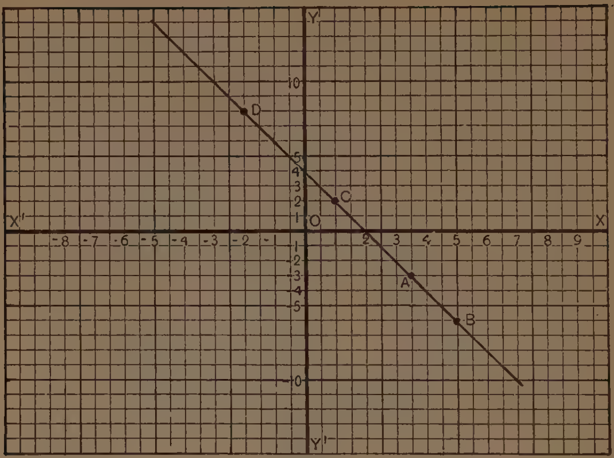


FIG. 6

EXERCISE 97

1. Determine the coördinates of *B*, *C*, and *D*. Determine whether they satisfy the equation of the graph?

2. Select any point not on the graph, find its coördinates, and determine whether they satisfy the equation of the graph.

3 Similarly, draw the graph of $y=x+3$ For x select the values -1, -2, -3, -4, $+1$, $+2$, $+3$, $+4$, 0. What does the graph appear to be?

4 Select three new points which are on the resulting graph, find their coördinates, and determine whether they satisfy the equation given in Example 3

5. Draw the graph of $\frac{y}{2}-\frac{x}{6}=1$.

Select at least four negative and four positive values of x, and from them determine the corresponding values of y. What sort of graph do you obtain?

6. Select any three points on the graph and determine whether their coördinates satisfy the equation.

153. The equations in the preceding paragraph have each had *two unknown numbers.* The following facts are to be remembered from the examples of § 152:

1. **The graph of an equation of the first degree (§ 77) having two unknowns is a straight line. For this reason, first degree equations are also called *linear equations.***

2. **The coördinates of every point on the line satisfy the equation.**

3. **The coördinates of every point not on the line do not satisfy the equation.**

154. A **Solution** of an equation having two or more unknowns is a set of values of the unknowns which satisfy the equation.

Be careful not to confuse the word "solution" in the sense of this paragraph with the same word when it is used to mark the process of solving an example as is done in the text.

EXAMPLE. Consider the equation $x + y = 5$.

$(1, 4)$ is a solution because $1 + 4 = 5$.

$(-8, +13)$ is a solution because $-8 + 13 = 5$.

155. Number of Solutions. There are an indefinitely large number of points upon a straight line. This is expressed by saying that there are an *infinite number* of points on a straight line. Since the coördinates of each point satisfy the equation of the line, then: *there are an infinite number of solutions of a linear equation with two unknowns.*

This fact is evident also because for every value of one of the unknowns, a value of the other may be found.

EXAMPLE. Consider the equation $2x + 3y = 15$.

When $x = 1$, $2 \cdot 1 + 3y = 15$ or $3y = 13$, $y = 4\frac{1}{3}$;

then $x = 1$, $y = 4\frac{1}{3}$ is one solution.

Similarly when,

$x = 2;\quad y = \frac{11}{3}.\qquad x = -3;\ y = +7.$

$x = 3;\quad y = 3.\qquad x = -5;\ y = +8\frac{1}{3}.$

Thus as x changes in value, in such an equation, y also changes in value, acquiring a new value. x and y are said to *vary*, and are called **Variables.** Hereafter these equations will be called *equations having two variables.*

An equation of the first degree having two variables has an infinite number of solutions.

Such equations are called **Indeterminate Equations.**

156. The graph of an indeterminate linear equation with two variables is always a straight line by § 153 A straight line may be drawn with a ruler as soon as two of its points are known. This leads to the

Rule. — To draw the graph of a linear equation having two variables:

1. Select one value for one variable and determine the corresponding value of the other variable; this gives one solution.

2. Determine a second solution as in step 1.

3. Plot the two points whose coördinates are the pairs of numbers and connect them with a straight line.

4. Check the result by finding a third solution, and plotting the corresponding point. The third point should fall upon the graph obtained in step 3.

EXAMPLE. Draw the graph of $4x - 3y = 6$.

SOLUTION: 1. Let $x = 0$; then $y = -2$

$$(4 \cdot 0 - 3y = 6; \ -3y = 6; \ y = -2.)$$

2. Let $x = 3$; then $y = 2$.

$$(4 \cdot 3 - 3y = 6; \ -3y = 6 - 12; \ -3y = -6; \ y = 2.)$$

3. Plot the points and draw the line. See Fig. 7.

CHECK: Let $x = 6$; then $y = 6$. Is this point on the line?

NOTE. To get the best results, it is necessary to use coordinate paper for the following exercises. Fair results may be obtained by using paper ruled by the pupil.

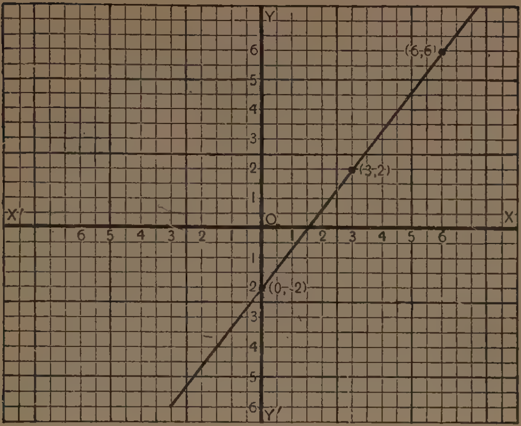


FIG. 7

EXERCISE 98

Draw the graphs of the following equations, each on a separate sheet of graph paper:

1. $x + y = 5$.
2. $2x - y = 6$.
3. $2x + 3y = 6$.
4. $3x - 2y = 12$.
5. $5x + 4y = 20$.
6. $3x - 5y = 15$.
7. $7x + 4y = 2$.
8. $5x - 3y = -6$.
9. $x - 6y = -10$.
10. $3x = 5 - 7y$.

157 Independent Equations and Simultaneous Equations.

EXAMPLE. Draw upon the same sheet the graph of

$$2x - y = 4. \qquad (1)$$
$$2x + 3y = 12. \qquad (2)$$

SOLUTION: 1. For equation 1: $2x - y = 4$.
If $x = 0$, $y = -4$; if $y = 0$, $x = +2$.
Solutions: $(0, -4)$ and $(+2, 0)$.

2. For equation 2 : $2x + 3y = 12$.
If $x = 0$, $y = 4$; if $y = 0$, $x = 6$.

Solutions : $(0, +4)$ and $(+6, 0)$.

3. See Fig. 8 for the graphs

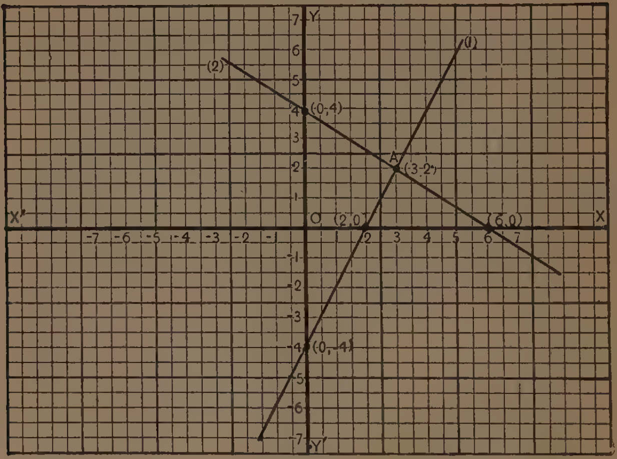


FIG. 8

QUESTIONS : 1. What is true about the coördinates of all points on line 1 ? (§ 153.)

2. What is true about the coördinates of all points on line 2 ?

3. What then is true of the coördinates of point A ?

4. Are there any other points whose coördinates will satisfy both equations ?

Two equations, each having two variables, are called **Independent Equations** when each has solutions which do not satisfy the other. Thus, the equations above are independent. Their graphs are two different straight lines.

Two linear equations which are independent and which have one common solution are called **Simultaneous Linear Equations.** Their graphs cross at one point. The above equations are simultaneous since they have the common solution (3, 2).

Rule. — To determine graphically the common solution of two simultaneous linear equations having two variables:

1. Draw upon one sheet the graphs of both equations

2. Find the coördinates of the point common to the two lines. This is the common solution.

3. Test the common solution by substituting it in both equations.

Historical Note. One of the notable advances in mathematics is intimately associated with the subject of this chapter. The idea of representing some geometrical figures by algebraic expressions occurred to mathematicians early. In the 14th century, Oresme, a French mathematician, discussed certain mathematical ideas by means of coördinates. He, however, used only positive coördinates, thus confining himself to what is called the first quadrant.

To the French mathematician, Descartes (1596–1650), is due the extension of the use of coördinates. He considered negative as well as positive coördinates and also discussed two or more lines, or curves, drawn with respect to the same system of coördinate axes. His improvement brought with it a better understanding of negative numbers and of negative roots of equations, and laid the foundation for Analytic Geometry, one of the subjects of great interest to mathematicians.

EXERCISE 99

Determine graphically which of the following pairs of equations are simultaneous, and which are *dependent;* if simultaneous, find their common solution:

1. $\begin{cases} a + b = 5. \\ a - b = 9. \end{cases}$

2. $\begin{cases} 2x + y = 7. \\ 2x - y = 5. \end{cases}$

3. $\begin{cases} x + 2y = 12. \\ 3x - y = 1. \end{cases}$

4. $\begin{cases} 3x - 2y = -12. \\ 4x + 2y = -2. \end{cases}$

5. $\begin{cases} 2x + y = 4. \\ 6x + 3y = 12. \end{cases}$

6. $\begin{cases} 5x + 2y = 10. \\ x - y = 9. \end{cases}$

7. $\begin{cases} 3a - 2b = 6. \\ 6a - 4b = 12. \end{cases}$

8. $\begin{cases} 5x - 6y = 8. \\ 9x - 4y = -6. \end{cases}$

9. Draw on one sheet the graphs of the following equations and thus determine whether they have a common solution:

(*a*) $5x-2y=7$; (*b*) $3x+2y=17$; (*c*) $x-3y=-9$.

10 Determine whether the following equations have a common solution:

(*a*) $2x-y=8$; (*b*) $2x-y=4$; (*c*) $2x+3y=20$.

158 Inconsistent Equations. Two equations may be independent (§ 157), and yet not be simultaneous; that is, they may not have a common solution. The graphs of two simultaneous linear equations are two intersecting straight lines.

Two independent linear equations which do not have a common solution are called **Inconsistent Equations.** Their graphs are parallel lines.

EXERCISE 100

Determine graphically which sets of equations are inconsistent; if simultaneous, determine the common solution.

1. $\begin{cases} x-3y=6. \\ 2x-6y=-18. \end{cases}$

2. $\begin{cases} 5a-2b=10. \\ 15a-6b=-24. \end{cases}$

3. $\begin{cases} 2x-3y=18 \\ x+3y=0. \end{cases}$

4. $\begin{cases} 4m-3n=12 \\ 16m-12n=24. \end{cases}$

5. $\begin{cases} 2x-5y=-16 \\ 3x+7y=5. \end{cases}$

6. $\begin{cases} 2x+3y=6. \\ 4x+6y=8. \end{cases}$

159. Sometimes it is difficult to find the common solution accurately by this graphical method. As an example, find the common solution of the pair of equations:

(1) $5x-9y=11$; (2) $3x+7y=9$.

In the next chapter, other methods of solving simultaneous linear equations are given.

XIII. SIMULTANEOUS LINEAR EQUATIONS

DEFINITIONS

160. If a rational and integral monomial (§ 113) involves wo or more letters, its degree *with respect to them* is denoted y the sum of their exponents.

Thus, $2a^2bxy^3$ is of the *fourth* degree with respect to x and y.

161. If each term of an equation containing one or more unnown numbers is rational and integral, the **Degree of the Equa-ion** is the degree of its term of highest degree.

Thus, if x and y represent unknown numbers,

$ax - by = c$ is an equation of the *first* degree ;
$x^2 + 4x = -2$ is an equation of the *second* degree ,
$2x^2 - 3xy^2 = 5$ is an equation of the *third* degree.

162. An equation of the first degree is also called **a Linear quation.**

163. A **Solution** of an equation with two unknowns is a set f values of the unknowns which together satisfy the equation.

Thus $x = 12$ and $y = 2$ is a solution of $x + 2y = 16$.

A linear equation with two unknowns has an *infinite* number f solutions. A solution is obtained by assigning any value to ne unknown and finding the corresponding value of the other nknown. Thus :

(*a*) Some of the solutions of the equation $x + 2y = 16$ are :

$x = +1$	$+2$	-4	$+10$	$+20$
$y = +7.5$	$+7$	$+10$	$+3$	-2

(*b*) Some of the solutions of the equation $x - 2y = 4$ are :

$x = +1$	$+2$	-4	$+10$	$+20$
$y = -1.5$	-1	-4	$+3$	$+8$

An equation with two unknowns is called an **Indeterminat** **Equation,** and the unknowns are called **Variables.**

164. Two linear equations, each containing two variable are said to be **Independent Equations** if each has solutions whic are not solutions of the other.

The two equations in § 163 are independent. Why?

The equations $x + y = 5$ and $2x + 2y = 10$ are not independent; fo the second equation can be reduced to the form of the first by dividin each term by 2; hence every solution of one equation is also a solutio of the other. They are called *dependent* equations.

165. Two independent linear equations having *one commo* *solution* are called **Simultaneous Equations.**

The two equations in § 163 are simultaneous. Why?

166. Two independent linear equations which do not hav any common solution are called **Inconsistent Equations.**

The equations $x + 2y = 16$ and $2x + 4y = 21$ are inconsistent

167. To **Solve** a *set* of simultaneous equations is to find the common solution.

Example 1. Solve the equations $\begin{cases} 5x - 3y = 19. & (\\ 7x + 4y = 2. & (\end{cases}$

Solution: 1. M_4* (1): $20x - 12y = 76.$ (

2. M_3 (2): $21x + 12y = 6.$ (

3. Add (3) and (4): $41x = 82.$ (

4. D_{41} (5): $x = 2.$ (

5. Substitute this value of x in (1): $10 - 3y = 19.$ (

$-3y = 9.$ (

$y = -3.$ (

6. The common solution is $x = 2, y = -3.$

Check: Substitute in (1): $10 + 9 = 19.$

Substitute in (2): $14 - 12 = 2.$

---

* See § 42 for the symbol M_4. Read this "multiply both members of eq tion (1) by 4."

Notice: 1. That, in equations (1) and (2), the coefficients of y are not the ame.

2. That, by multiplications, the coefficients of y are made the same in bsolute value in equations (3) and (4).

3. That, by addition, $12y$ disappears in equation (5), and that $20x$ and $21x$ re combined, giving $41x$. This is allowable because we assume that x and y epresent the same numbers respectively in all of the equations, namely, the articular numbers which together form the common solution of equations 1) and (2). y is said to be **Eliminated.**

Elimination means to cause to disappear; thus y was made to disappear y adding the two equations (3) and (4).

4. That the remaining number, x, is then easily found.

The solution is an example of **Elimination by Addition.**

Example 2. Solve the equations

$$\begin{cases} 15a + 8b = 1. & (1) \\ 10a - 7b = -24. & (2) \end{cases}$$

Solution: 1. M_2 (1): $30a + 16b = 2.$ (3)

2. M_3 (2): $30a - 21b = -72.$ (4)

3. Subtract (4) from (3): $37b = 74.$ (5)

4. D_{37} (5): $b = 2.$ (6)

5. Substitute the value of b in (1): $15a + 16 = 1.$ (7)

$\therefore 15a = -15.$ (8)

$\therefore a = -1.$ (9)

The solution is: $a = -1;\ b = +2.$

Substitute the values of a and b in equations (1) and (2).

The solution is an example of **Elimination by Subtraction.**

Rule. — To solve two simultaneous linear equations having two variables by the addition or subtraction method of elimination:

1. Multiply, if necessary, both the first and second equations by such numbers as will make the coefficients of one of the variables of equal absolute value.

2. If the coefficients have the same sign, subtract one equation from the other; if they have opposite signs, add the equations.

3 Solve the equation resulting from step 2 for the other variable.

4 Substitute the value of the variable found in step 3 in any equation containing both variables, and solve for the remaining variable

5. Check the solution by substituting it in both of the original equations.

Historical Note. Little progress was made in solving linear equations having more than one unknown until the latter part of the 15th century, although mathematicians before that time had considered such problems. After Stifel and Stevin had introduced somewhat simple notations for several unknowns and their powers, definite methods for solving equations of the first degree with two unknowns were developed. Johannes Buteo, a French monk, (1492–1572), solved equations with three unknowns in a text on algebra which appeared in 1559.

EXERCISE 101

Solve by the method of addition or subtraction:

1. $\begin{cases} 3x + y = 11. \\ 5x - y = 13. \end{cases}$

2. $\begin{cases} 4a + 3b = -1. \\ 5a + b = 7. \end{cases}$

3. $\begin{cases} r - 6s = -10. \\ 2r - 7s = -15. \end{cases}$

4. $\begin{cases} 5m + 4n = 22. \\ 3m + n = 9. \end{cases}$

5. $\begin{cases} 7c - 2d = 31. \\ 4c - 3d = 27. \end{cases}$

6. $\begin{cases} 5a + 3b = -9. \\ 3a - 4b = -17. \end{cases}$

7. $\begin{cases} 6x + 2y = -3. \\ 5x - 3y = -6. \end{cases}$

8. $\begin{cases} 3s + 7t = 4. \\ 7s + 8t = 26. \end{cases}$

9. $\begin{cases} 11p - 5q = 4. \\ 8p - 6q = 10. \end{cases}$

10. $\begin{cases} 5r - 9y = 1. \\ 8r - 10y = -5. \end{cases}$

11. $\begin{cases} 8c + 9t = 3. \\ 8c - 9t = 77. \end{cases}$

12. ✓ $\begin{cases} 6x - 11y = -4. \\ 15x + 4y = 53. \end{cases}$

13. $\begin{cases} 3x + 7y = 2. \\ 7x + 8y = -2. \end{cases}$

14. $\begin{cases} 4v + 15g = 7. \\ 14v + 6g = 9. \end{cases}$

15. ✓ $\begin{cases} 28x - 36y = 1. \\ 14x + 15y = 6. \end{cases}$

16. $\begin{cases} 7x + 9y = 8. \\ 9x - 8y = 69. \end{cases}$

17. ✓ $\begin{cases} 2x - \frac{5}{2}y = 13. \\ \frac{x}{3} + \frac{y}{5} = \frac{14}{15}. \end{cases}$

18. $\begin{cases} \frac{7}{6}r + 2 = \frac{s}{2}. \\ \frac{s}{4} - \frac{r}{3} = 1\frac{3}{4}. \end{cases}$

19. ✓ $\begin{cases} \frac{3}{8}m + n = \frac{1}{4}. \\ \frac{m}{2} - \frac{5n}{7} = 1\frac{5}{14}. \end{cases}$

20. ✓ $\begin{cases} \frac{a}{4} + \frac{b}{5} = 1. \\ \frac{2a}{9} + 2 = \frac{b}{9}. \end{cases}$

168. Elimination by Substitution is a second common method of solving simultaneous equations.

EXAMPLE. Solve the equations $\begin{cases} 7x - 9y = 15. & (1) \\ 8y - 5x = -17. & (2) \end{cases}$

SOLUTION: 1. Solve (1) for x in terms of y:

$$7x = 15 + 9y;\ x = \left(\frac{15 + 9y}{7}\right). \qquad (3)$$

2. Substitute this value of x in (2):

$$8y - 5\left(\frac{15 + 9y}{7}\right) = -17. \qquad (4)$$

3. M_7 (4): $56y - 5(15 + 9y) = -119.$ (5)

4. Expanding: $56y - 75 - 45y = -119.$ (6)

5. Combining: $11y - 75 = -119.$ (7)

6. A_{75}: $11y = -44.$

7. D_{11}: $y = -4.$

8. Substituting the value of y in (3):

$$x = \frac{15 + 9(-4)}{7} = \frac{15 - 36}{7} = \frac{-21}{7} = -3.$$

9. The solution is: $x = -3$, $y = -4$. Check it by substitution in equations (1) and (2).

NOTE. In step 3, when multiplying $5\left(\frac{15+9y}{7}\right)$ by 7, one obtains $7 \cdot \frac{5(15+9y)}{7}$. The 7's cancel, giving the result $5(15+9y)$.

Rule. — **To solve two simultaneous linear equations having two variables by the substitution method of elimination:**

1. Solve one equation for one variable in terms of the other variable.

2. Substitute for this variable in the other equation the value found for it in step 1.

3. Solve the equation resulting in step 2 for the second variable.

4. Substitute the value of the second variable, obtained in step 3, in any equation containing both variables and solve for the first variable.

5. Check the solution by substituting it in the original equations.

Historical Note. The earliest use of the Substitution Method of Elimination in print, of which we have any record, is in Newton's Arithmetica Universalis, in 1707.

EXERCISE 102

Solve by the substitution method:

1. $\begin{cases} 3m+2n=17. \\ 4m+n=16. \end{cases}$

2. $\begin{cases} r-6s=2. \\ 3s-8r=29. \end{cases}$

3. $\begin{cases} 5x+8y=+2. \\ 10x-12y=+32. \end{cases}$

4. $\begin{cases} 2x+5y=13. \\ 7x-4y=-19. \end{cases}$

5. $\begin{cases} 8x+5y=18. \\ 16x-3y=10. \end{cases}$

6. $\begin{cases} 8p+5q=5. \\ 3p-2q=29. \end{cases}$

7. $\begin{cases} 5m-12n=-31. \\ 3m+22n=-4. \end{cases}$

8. $\begin{cases} 2x-3y=-14. \\ 3x+7y=48. \end{cases}$

9. $\begin{cases} 5x+9y=8. \\ 6y-9x=-7. \end{cases}$

10. $\begin{cases} 7r-6t=63. \\ 9r+2t=13. \end{cases}$

11. $\begin{cases} 8p+5q=-6. \\ 12p+10q=-5. \end{cases}$

12. $\begin{cases} 6w-9x=19. \\ 15w+7x=-41. \end{cases}$

13. $\begin{cases} 15A-12B=-54. \\ 10A+9B=-2. \end{cases}$

14. $\begin{cases} 7M+8N=19. \\ 9M-6N=57. \end{cases}$

15. $\begin{cases} 8p-3q=-1. \\ 4p+6q=+7. \end{cases}$

16. $\begin{cases} 6x-10y=5. \\ 15y-14x=-15. \end{cases}$

17. $\begin{cases} 9c+8d=-6. \\ 12c+10d=-7. \end{cases}$

18. $\begin{cases} 3e+7f=-23. \\ 5e+4f=-23. \end{cases}$

19. $\begin{cases} 7g+8k=-10. \\ 11g+6k=-19. \end{cases}$

20. $\begin{cases} 5r-8s=60. \\ 6r+7s=-11. \end{cases}$

169. As a rule, the equations containing two variables do not occur in as simple form as those given in §§ 167, 168.

Example. Solve the equations

$$\begin{cases} \dfrac{7}{x+3}-\dfrac{3}{y+4}=0. & (1) \\ x(y-2)-y(x-5)=-13. & (2) \end{cases}$$

Solution: 1. Simplify equation (1):

$$\frac{7}{x+3}-\frac{3}{y+4}=0.$$

$$\therefore 7(y+4)-3(x+3)=0.$$

$$3x-7y=19. \qquad (3)$$

2. Simplify equation (2):

$$x(y-2)-y(x-5)=-13.$$

$$2x-5y=+13. \qquad (4)$$

3. The equations (1) and (2) thus become in (3) and (4):

$$3x-7y=19. \qquad (3)$$

$$2x-5y=13. \qquad (4)$$

Solve these equations by either of the two methods of elimination; the solution will be found to be $x=4$, $y=-1$.

Check: Substitute the values of x and y in equations (1) and (2).

Equations (3) and (4) are called the **Standard Form** of equations (1) and (2).

Rule. — To solve complicated simultaneous linear equations having two variables:

1. Reduce the equations to the standard form by clearing of fractions and simplifying.

2. Solve the resulting equations by either of the two methods.

3. Check by substituting in the original equations.

EXERCISE 103

Reduce to the standard form and solve:

1.✓ $\begin{cases} \dfrac{2x}{3}+\dfrac{3y}{4}=\dfrac{-7}{2}. \\ \dfrac{x}{4}-\dfrac{2y}{5}=\dfrac{11}{2}. \end{cases}$

2. $\begin{cases} 8p+7q=12. \\ \dfrac{p+2q}{4}+\dfrac{2p+q}{3}=1. \end{cases}$

3. $\begin{cases} 10m-\dfrac{y-5}{7}=11. \\ 8y-\dfrac{m+3}{4}=-17. \end{cases}$

4.✓ $\begin{cases} \dfrac{r}{3}-\dfrac{s}{2}=1. \\ \dfrac{3+2r}{5}-\dfrac{1+5s}{11}=+2 \end{cases}$

5. $$\begin{cases} \dfrac{2w}{3} + \dfrac{3t}{4} = \dfrac{1}{6}. \\ 5t - 8w = \dfrac{22}{3}. \end{cases}$$

6. $$\begin{cases} \dfrac{r+t}{2} - \dfrac{r-t}{3} = 8. \\ \dfrac{r+t}{3} + \dfrac{r-t}{4} = 11. \end{cases}$$

7. $$\begin{cases} \dfrac{c+d-2}{c-d} = -\dfrac{1}{3}. \\ \dfrac{3c+d-3}{2d-c} = -\dfrac{1}{11}. \end{cases}$$

8. $$\begin{cases} \dfrac{3}{x-1} + \dfrac{4}{y-1} = 0. \\ \dfrac{5}{2x-3} - \dfrac{7}{2y+13} = 0. \end{cases}$$

9. $$\begin{cases} \dfrac{5}{m-2} + \dfrac{7}{2n-3} = 0. \\ \dfrac{1}{2m+5} + \dfrac{8}{3n-7} = 0. \end{cases}$$

10. $$\begin{cases} \dfrac{6+a-b}{1-a-b} = -\dfrac{7}{4}. \\ 2a + 3b = -1. \end{cases}$$

11. $$\begin{cases} \dfrac{x+5y}{13} - \dfrac{2y+x}{11} = -1. \\ 3x - y = 2. \end{cases}$$

12. $$\begin{cases} \dfrac{y}{3} - \dfrac{x}{2} = 2. \\ \dfrac{3-2x}{5} - \dfrac{4+5y}{11} = 4. \end{cases}$$

13. $$\begin{cases} (m+1)(n+9) - (m+5)(n-7) = 112. \\ 2m + 3n + 9 = 0. \end{cases}$$

14. $$\begin{cases} \dfrac{r-s}{2} = \dfrac{25}{6} - \dfrac{r+s}{3}. \\ \dfrac{r+s-9}{2} - \dfrac{s-r-6}{3} = 0. \end{cases}$$

15. $$\begin{cases} \dfrac{w-2}{5} - \dfrac{10-w}{3} - \dfrac{p-10}{4} = 0. \\ \dfrac{p+2}{6} - \dfrac{2w+p}{32} - \dfrac{w+13}{16} = 0. \end{cases}$$

16. $$\begin{cases} \dfrac{x-y}{3} - \dfrac{2x+y}{2} = 0. \\ \dfrac{x+2y}{2} - \dfrac{x}{4} = -\dfrac{11}{4}. \end{cases}$$

17. $$\begin{cases} \dfrac{4a+b}{5} - \dfrac{6a-3b}{3} = -3. \\ \dfrac{8a+5b}{2} - \dfrac{10a-b}{7} = -4. \end{cases}$$

170. Certain equations in which the variables occur in the denominators of fractions may be solved readily without clearing the equations of fractions.

EXAMPLE. Solve the equations
$$\begin{cases} \dfrac{10}{x} - \dfrac{9}{y} = 8. & (1) \\ \dfrac{8}{x} + \dfrac{15}{y} = -1. & (2) \end{cases}$$

SOLUTION: Eliminate the term containing y:

1. M_5 (1): $\dfrac{50}{x} - \dfrac{45}{y} = 40.$ (3)

2. M_3 (2): $\dfrac{24}{x} + \dfrac{45}{y} = -3.$ (4)

3. Add (4) and (3): $\dfrac{74}{x} = 37.$ (5)

$\therefore 74 = 37x$ or $x = 2.$ (6)

4. Substitute 2 for x in (1): $5 - \dfrac{9}{y} = 8.$ (7)

5. Solve (7) for y: $y = -3.$

CHECK: Substitute $x = 2$, $y = -3$ in equations (1) and (2).

EXERCISE 104

90

Solve the following sets of equations:

1. $\begin{cases} \dfrac{9}{a} + \dfrac{10}{b} = -1. \\ \dfrac{6}{a} + \dfrac{15}{b} = +1. \end{cases}$

2. $\begin{cases} \dfrac{10}{c} - \dfrac{9}{d} = 4. \\ \dfrac{8}{c} - \dfrac{15}{d} = \dfrac{9}{2}. \end{cases}$

3. $\begin{cases} 2p + \dfrac{5}{q} = -11. \\ 4p - \dfrac{3}{q} = \dfrac{21}{2}. \end{cases}$

4. ✓ $\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5. \\ \dfrac{1}{x} - \dfrac{1}{y} = 1. \end{cases}$

5. ✓ $\begin{cases} \dfrac{3}{r} - \dfrac{1}{s} = 9. \\ \dfrac{4}{r} + \dfrac{3}{s} = -1. \end{cases}$

6. ✓ $\begin{cases} \dfrac{6}{w} + \dfrac{4}{v} = \dfrac{4}{5}. \\ \dfrac{9}{w} + \dfrac{5}{v} = \dfrac{7}{10}. \end{cases}$

7. $\begin{cases} \dfrac{2}{A} - \dfrac{3}{B} = -1. \\ \dfrac{5}{A} - \dfrac{6}{B} = -\dfrac{1}{2}. \end{cases}$

8. $\begin{cases} \dfrac{4}{r} - \dfrac{3}{t} = 17. \\ \dfrac{5}{r} + \dfrac{6}{t} = -\dfrac{3}{2}. \end{cases}$

9. $\begin{cases} \dfrac{5}{3x} - \dfrac{7}{y} = \dfrac{29}{9}. \\ \dfrac{3}{x} + \dfrac{5}{4y} = -\dfrac{9}{8}. \end{cases}$

10. $\begin{cases} \dfrac{5}{2x} - \dfrac{4}{3y} = \dfrac{1}{2}. \\ \dfrac{2}{3x} - \dfrac{1}{2y} = \dfrac{7}{72}. \end{cases}$

171. In solving problems where two or more letters are used to represent unknown numbers, *as many independent equations* must be obtained from the conditions of the problem *as there are letters used.*

Example 1. Four seventeenths of the greater of two numbers exceeds the less number by 5; if the greater be divided by the less, the quotient is 5 and the remainder 10. Find the two numbers.

Solution: 1. Let g = the greater number,

and l = the less number.

2. Then, $\frac{4}{17} g = l + 5.$ (1)

3. When the greater is divided by the less, the quotient is 5 and the remainder 10; therefore,

$$g = 5l + 10. \qquad (2)$$

4. Solving equations (1) and (2), $g = 85$, $l = 15$.

Check: $\frac{4}{17}$ of $85 = 20$; $20 - 15 = 5$,

and $85 = 5 \cdot 15 + 10.$

Note. In equation (2), use is made of the fact that the dividend equals the divisor times the quotient plus the remainder. Thus, when 17 is divided by 2,

$$17 = 8 \cdot 2 + 1$$

Example 2. If 3 be added to both numerator and denominator of a fraction, its value becomes $\frac{2}{3}$; and if 2 be subtracted from both numerator and denominator of the fraction its value becomes $\frac{1}{2}$. Required the fraction.

Solution: 1. Let $n =$ the numerator,

and $d =$ the denominator.

$$\therefore \frac{n}{d} = \text{the fraction.}$$

2. By the first condition: $\dfrac{n+3}{d+3} = \dfrac{2}{3}$. (1)

By the second condition: $\dfrac{n-2}{d-2} = \dfrac{1}{2}$. (2)

63
54
9

3. Solving the equations (1) and (2), $n = 7$, $d = 12$.

Therefore the fraction is $\frac{7}{12}$.

Check: $\dfrac{7+3}{12+3} = \dfrac{10}{15} = \dfrac{2}{3}$; $\dfrac{7-2}{12-2} = \dfrac{5}{10} = \dfrac{1}{2}$

Note. Check the solution by going back to the conditions of the problem.

EXERCISE 105

1. Divide 59 into two parts such that two thirds of the less shall be less by 4 than four sevenths of the greater.

2. Find two numbers such that two fifths of the greater exceeds one half of the less by 2, and four thirds of the less exceeds three fourths of the greater by 1

3. If 5 be added to the numerator of a certain fraction, the value of the fraction becomes $\frac{5}{3}$; and if 5 be subtracted from its denominator, the value of the fraction becomes $\frac{5}{2}$. Find the fraction.

4. If 9 be added to both terms of a fraction, its value becomes $\frac{6}{7}$; and if 7 be subtracted from both terms of the fraction, its value becomes $\frac{2}{3}$. Find the fraction.

5. In 1910, the cost of 3 tons of anthracite coal in Philadelphia exceeded the cost of 4 tons of bituminous coal in Baltimore by $3.10; and the cost of 9 tons of the bituminous coal exceeded the cost of 5 tons of the anthracite by 90¢. Find the cost of the anthracite and of the bituminous coal in 1910.

6. A's age is three fifths of B's age; but in 16 years A's age will be five sevenths of B's age. Find their ages at present.

$\frac{3b}{5} = a$

$5(b+16) = a+16$

7. If twice the greater of two numbers be divided by the less, the quotient is 3 and the remainder is 7; if five times the less be divided by the greater, the quotient is 2 and the remainder is 23. Find the numbers.

8. On the tower of the City Hall of Philadelphia is a statue of William Penn. If the total height of the tower and statue be divided by the height of the statue, the quotient is 14 and the remainder is 29; the height of the statue exceeds $\frac{1}{17}$ of the height of the tower by 7 feet. Find the height of the tower and of the statue.

9. If the numerator of a fraction be trebled, and the denominator be increased by 8, the value of the fraction becomes $\frac{9}{8}$; and if the denominator be halved, and the numerator be decreased by 7, the fraction becomes $\frac{1}{4}$. Find the fraction.

10. The City Hall of Philadelphia is said to cover a greater area than any other building in the United States. One fifth of its width exceeds one sixth of its length by 13 feet; and one ninth of its length exceeds one tenth of its width by 7 feet. Find its dimensions.

11. The perimeter of a certain isosceles triangle (see p. 102) is 140 inches. The side exceeds the base by 10 inches. Find the three sides of the triangle.

12. Three years ago A's age was $\frac{4}{3}$ of B's age; but in nine years his age will be $\frac{11}{9}$ of B's age. Find their present ages.

13. If the age of a university is reckoned from the date of its founding, then, in 1912, the age of Yale exceeded the age of Princeton by 46 years. Four years later, one half of the age of Princeton will exceed one third of the age of Yale by 13 years. Find when each was founded.

14. Twice the shorter side of a parallelogram exceeds the longer side by 5 inches; one third of the sum of the shorter side and 9 exceeds one fifth of the longer side by 3. Find the sides of the parallelogram.

15 In 1912 the sum of the ages of the oldest English university, Oxford, and the oldest American university, Harvard, was 1316 years; the age of Oxford exceeded three and three fourths times the age of Harvard by 5 years. Find when each was founded.

16. If one weight is placed 8 inches from the fulcrum of a lever, it balances another weight which is 6 inches from the fulcrum; if the first weight be decreased by 3 pounds, and the second be increased by 4 pounds, the resulting weights will balance if placed 10 feet and 5 feet respectively from the fulcrum Find the two weights (§ 143).

17 The sum of the reciprocals of two numbers is $\frac{12}{35}$ Twice the reciprocal of the greater number exceeds the reciprocal of the less number by $\frac{3}{35}$. Find the two numbers.

(Hint. The reciprocal of 3 is $\frac{1}{3}$; of a is $\frac{1}{a}$.)

18. The sum of the reciprocals of two numbers is $\frac{11}{4}$. If twice the reciprocal of the less be increased by four times the reciprocal of the greater, the sum is 8 Find the two numbers.

19 A purse contained $6.55 in quarters and dimes; after 6 quarters and 8 dimes had been taken out, the number of quarters equalled three times the number of dimes. How many of each kind of coin were there?

20. In 7 years, A will be three times as old as B, and 8 years ago he was 6 times as old What are their present ages?

21. The length of a room exceeds its width by 4 feet. If 2 feet are added to the length, and 3 feet to the width, the area is increased by 103 square feet. Find the dimensions.

Solution: 1. Let l = the number of feet in the length,
and w = the number of feet in the width.
$\therefore lw$ = the area.

2. $$l = w + 4. \qquad (1)$$

3. $l + 2 =$ the new length.
$w + 3 =$ the new width.
$\therefore (l + 2)(w + 3) =$ the new area.
4. $\therefore (l + 2)(w + 3) = lw + 103.$ (2)
5. Solve the equations (1) and (2), and check.

22. If a rectangular lot were 6 feet longer and 5 feet wider than it is now, it would contain 839 square feet more; if it were 4 feet longer, and 7 feet wider, it would contain 879 square feet more. Find its length and width.

23. If one weight, increased by 10 pounds, be placed 6 feet from the fulcrum, it will balance a second weight, placed $4\frac{1}{2}$ feet from the fulcrum; if the second weight, increased by 10 pounds, be placed $3\frac{1}{2}$ feet from the fulcrum, it will balance the first weight placed 6 feet from the fulcrum. Find the two weights.

24. Will weighs 50 pounds. When Will seats himself 6 feet from the fulcrum and John seats himself 4 feet from the fulcrum on the same side, they exactly balance James, who is sitting $7\frac{1}{4}$ feet from the fulcrum on the other side. When Will and John change places, they find that James must sit $7\frac{3}{4}$ feet from the fulcrum. How much do John and James weigh?

25. A crew can row 10 miles down stream in 50 minutes, and 12 miles up stream in an hour and a half. Find the rate in miles an hour of the current, and of the crew in still water.

(HINT. Review *River Problems*, Exercise 90, § 144; express 50 minutes as $\frac{5}{6}$ hour.)

26. A motor boat which can run at the rate of 15 miles an hour in still water went down stream a certain distance in 4 hours. It took 6 hours for the trip back. What was the distance and the rate of the current?

27. A crew are rowing on a stream the rate of whose current is known to be 2 miles an hour; they find that it takes them

one and one third hours to go down, and four hours to come back a certain distance. Find the distance and the rate of the crew in still water.

28. An express train travels 30 miles in 27 minutes less time than a slow train. If the rate of the express train were $\frac{5}{4}$ as great, and if the rate of the slow train were $\frac{4}{5}$ as great, the express train would travel 30 miles in 54 minutes less time than the slow train. Find the rate of each train in miles an hour.

29. If a field is made 5 feet longer and 7 feet wider, its area would be increased by 830 square feet; but if its length is made 8 feet less, and its width 4 feet less, its area is diminished by 700 square feet. Find its length and width.

30. The fore wheel of a carriage makes 8 revolutions more than the hind wheel in going 180 feet; but if the circumference of the fore wheel were $\frac{4}{3}$ as great, and of the hind wheel $\frac{6}{5}$ as great, the fore wheel would make only 5 revolutions more than the hind wheel in going the same distance. Find the circumference of each wheel.

31. A man has $2500 invested from which he receives a total income of $135. Part of the money is invested at 6 % and part at $4\frac{1}{2}$ %. How much is invested in each way?

32. A man has altogether $5000 of savings. He has part invested in a 5 % bond, and the balance invested in a mortgage drawing 6 %. If his total income is $280, how much has he invested in each way?

33. A man has $1200 invested at one rate of interest and $500 at a rate which is 1 per cent greater than the former rate The income from the first investment exceeds the income from the second investment by $23. Find the rate at which each sum is invested.

34. The simple interest on $800 at 5 % for a certain number of years exceeds the simple interest on $300 at 6 % for a

second period of years by \$60. If the second period of years exceeds the first by 4 years, find the number of years each sum is invested.

35. There are two supplementary angles such that $\frac{1}{3}$ of the larger exceeds $\frac{2}{5}$ of the smaller by 5°. Find the angles (§ 13.)

36. One angle of a triangle is 35°. If the number of degrees in one of the remaining two angles is divided by the number in the other, the quotient is 9 and the remainder is 10. Find the three angles of the triangle. (See § 13.)

37. The fastest train on the Pennsylvania R. R. makes the trip between Chicago and Fort Wayne, Indiana, a distance of 148 miles, in 1 hour and 20 minutes less time than one of the ordinary trains. Its rate is $\frac{3}{2}$ that of the ordinary train. Find the rate of each train.

38. A and B working together can do a piece of work in 6 days; they can also complete the work if A works 10 days and B 3 days. How many days would it take each of them to do the work alone? (See § 142.)

39. A and B working together can do a piece of work in $7\frac{1}{5}$ days. If A works alone for 3 days, and B alone for 6 days, they would complete $\frac{7}{12}$ of the work. Find how long it would take each alone to do the work.

40. A man has a sum of money invested at a certain rate of interest. Another man has a sum greater by \$3000, invested at a rate 1 % less, and his income is \$45 less than that of the first. A third man has a sum less by \$2000 than that of the first, invested at a rate 1 % greater, and his income is \$40 greater than that of the first. Find the capital of each man, and the rate at which it is invested.

172. Relations between the Digits of a Number. Integral numbers are written by means of the *digits*.

Thus, 372 is a number of three digits; 3 represents 300 units, 7 represents 70 units, and 2 represents 2 units.

Similarly, if t is the *tens' digit* and u is the *units' digit*, the number is $10\,t + u$.

When the digits of a number are *reversed*, a new number is formed; thus, reversing 52 gives 25. Notice that

$$52 = 10 \times 5 + 2 \text{ and } 25 = 10 \times 2 + 5.$$

Similarly, if x and y are the tens and units digits of a number, the number is $10\,x + y$; if the digits are reversed, the new tens' digit is y, the new units' digit is x, and the new number is $10\,y + x$.

EXERCISE 106

1. Write the number whose units' digit is a, tens' digit b, and hundreds' digit c.

2. Write the number represented by reversing the digits in Example 1.

3. Representing the tens' digit of a number by t, and the units' digit by u represent:

(*a*) the sum of the digits;

(*b*) the number;

(*c*) the product of the digits;

(*d*) the quotient when the number is divided by the sum of the digits.

4. Representing the tens' digit of a number by x, and the units' digit by y:

(*a*) express the original number;

(*b*) express the number obtained by reversing the digits;

(*c*) express the quotient of the new number divided by the old number;

(*d*) express by an equation the fact that when the original number is divided by the sum of its digits the quotient is 5 and the remainder is 3. (See note, Example 1, § 171.)

180
5

5. The sum of the two digits of a number of two digits is 11; if the digits be reversed, the quotient of the new number divided by the old is 2 and the remainder is 7. Find the number

SOLUTION: 1. Let $t =$ the tens' digit,

and $u =$ the units' digit.

$$\therefore t + u = 11. \qquad (1)$$

2. $10t + u =$ the original number,

and $10u + t =$ the new number.

$$\therefore 10u + t = 2(10t + u) + 7. \qquad (2)$$

3. Solving the pair of equations, (1) and (2), gives $t = 3$, $u = 8$.

$\therefore$ the number is 38.

CHECK: The sum of the digits is 11.

$83 \div 38$ gives the quotient 2 and the remainder 7.

6. The tens' digit of a number exceeds its units' digit by 4. If the digits be reversed, the new number is 6 more than one half of the old number. Find the number

7. The sum of the two digits of a number is 9 If the digits be reversed, the quotient of the new number divided by the units' digit of the given number is 13 and the remainder is 1. Find the number.

8 The sum of the two digits of a number is 16; and if 18 be subtracted from the number, the remainder equals the number obtained by reversing the digits. Find the number.

9. If the digits of a number of two figures be reversed, the sum of the resulting number and twice the given number is 204; and if the given number is divided by the sum of its digits, the quotient is 7 and the remainder is 6. Find the number.

10. If a certain number be divided by the sum of its two digits, the quotient is 4 and the remainder is 3. If the digits be reversed, the sum of the resulting number and 23 is twice the given number. Find the given number.

a. b. c. = constants

173. Literal Simultaneous Equations. In solving literal simultaneous equations, the addition or subtraction method of elimination is usually the best.

Example. Solve for x and y the equations:

$$ax + by = c. \qquad (1)$$

$$rx + sy = t. \qquad (2)$$

Solution: Eliminate y.

1. M_s (1): $\quad sax + sby = sc. \qquad (3)$
2. M_b (2): $\quad brx + bsy = bt. \qquad (4)$
3. Subtract (4) from (3): $\quad sax - brx = sc - bt. \qquad (5)$

$$\therefore (sa - br)x = (sc - bt). \qquad (6)$$

$$\therefore x = \frac{(sc - bt)}{(sa - br)}.$$

Now, going back to (1) and (2), eliminate x.

4. M_r (1): $\quad rax + rby = rc. \qquad (7)$
5. M_a (2): $\quad rax + say = ta. \qquad (8)$
6. Subtract (8) from (7): $\quad (rb - sa)y = (rc - ta).$

$$\therefore y = \frac{(rc - ta)}{(rb - sa)}.$$

The solution is: $\quad x = \dfrac{sc - bt}{sa - br}; \quad y = \dfrac{rc - ta}{rb - sa} = \dfrac{ta - rc}{sa - br}.$

Note. In step 6, when subtracting say from rby, we notice that these are *like terms* since they have the common factor y, and then subtract by multiplying that common factor y by the difference of its coefficients, $(rb - sa)$. Or, we may think of the difference as being $rby - say$, which, factored, becomes $(rb - sa)y$.

EXERCISE 107

1. $\begin{cases} 3x + 4y = 7a. \\ 2x - 5y = 6b. \end{cases}$
2. $\begin{cases} 3x + ay = 5. \\ 2x - by = 6. \end{cases}$
3. $\begin{cases} 4x + my = n. \\ x + py = q. \end{cases}$
4. $\begin{cases} 2ax + y = b. \\ ax - 2y = c. \end{cases}$
5. $\begin{cases} mx + ny = p. \\ cx + dy = e. \end{cases}$
6. $\begin{cases} ax + by = a^2 + 3ab. \\ 2ax - 3by = 2a^2 - 4ab. \end{cases}$

[illegible]

7. $$\begin{cases} x + ay = ab - 2a. \\ bx + ay = -ab. \end{cases}$$

8. $$\begin{cases} ax - by = 2ab. \\ 2bx + 2ay = 3b^2 - a^2. \end{cases}$$

9. $$\begin{cases} bx + ay = a^2 + ab. \\ x - y = a + b. \end{cases}$$

10. $$\begin{cases} 2cx + dy = 2c^2 + d^2. \\ \dfrac{x+y}{c} = 3. \end{cases}$$

11. $$\begin{cases} \dfrac{a}{x} + \dfrac{b}{y} = a^2 + b^2. \\ \dfrac{a}{x} - \dfrac{b}{y} = a^2 - b^2. \end{cases}$$

12. $$\begin{cases} rx + sy = 0. \\ sx + ry = s^2 - r^2. \end{cases}$$

13. $$\begin{cases} mx - ny = 0. \\ x - y = \dfrac{m^2 - n^2}{mn}. \end{cases}$$

14. $$\begin{cases} x + ay = b. \\ bx + y = a. \end{cases}$$

15. Find two numbers whose sum is m and whose difference is n.

16. Divide the number c into two parts so that when the larger is divided by the smaller the quotient is d.

17. Divide the number r into two parts such that when the larger is divided by the smaller the quotient is s and the remainder is t.

18. A and B can do a piece of work together in k days; if A works 3 days and B works 5 days they can do $\frac{1}{2}$ of it. Find how long it would take each alone to do the piece of work.

19. The sum of the digits of a number of two digits is a; the number itself equals b times the units digit. Find the digits of the number.

20. If a be added to the numerator and b be added to the denominator of a certain fraction, its value becomes 1; if b be added to the numerator and a to the denominator, its value becomes 2. Find the fraction.

21. Find two numbers whose sum is c and such that b times the first exceeds a times the second by d.

22. Find two numbers such that the quotient of a divided by the greater exceeds by c the quotient of b divided by the

less; and such that the quotient of b divided by the greater exceeds by d the quotient of a divided by the less.

The following two sets of equations arise in the solution of problems in applied mathematics; find T and f:

23. $\begin{cases} mg - T = mf. \\ T = nf. \end{cases}$ **24.** $\begin{cases} mg - T = mf. \\ T - ng = nf. \end{cases}$

EQUATIONS CONTAINING THREE VARIABLES

174. In the preceding paragraphs equations having two variables have been solved; in each case two equations were given. It is interesting to study equations with more than two variables. For three variables, three equations are necessary.

Example. Solve the set of equations,

$$\begin{cases} 2x - y + z = 5. & (1) \\ 3x + 2y + 3z = 7. & (2) \\ 4x - 3y - 5z = -3. & (3) \end{cases}$$

Solution: Eliminate z by combining (1) and (2); the resulting equation will contain only x and y.

1. M_3 (1): $6x - 3y + 3z = 15.$ (4)
2. Subtract (2) from (4): $3x - 5y = 8.$ (5)

Now eliminate z by combining (1) and (3); the resulting equation will contain only x and y.

3. M_5 (1): $10x - 5y + 5z = 25.$ (6)
4. Add (3) and (6): $14x - 8y = 22.$ (7)

This gives the equations:

$$3x - 5y = 8. \qquad (5)$$
$$14x - 8y = 22. \qquad (7)$$

Solve this set of equations for x and y.

The solution is, $x = 1,\ y = -1.$

Substitute these values of x and y in (1) and obtain z.

$$2 + 1 + z = 5.$$
$$\therefore z = 2.$$

The complete solution is $x = 1,\ y = -1,\ z = 2.$

Check the solution by substituting it in each of the given equations.

10t + w =

var
then combine 1st or then

EXERCISE 108

1. $\begin{cases} a-2b+c=0. \\ a-b+2c=-11. \\ 2a-b+c=-9. \end{cases}$

2. $\begin{cases} 3x-2y=1. \\ 3x+4z=5. \\ 5z+3y=4. \end{cases}$

3. $\begin{cases} 12m-4n+p=3. \\ m-n-2p=-1. \\ 5m-2n=0. \end{cases}$

4. $\begin{cases} 5r-3s-4t=-1. \\ 2s+3t=+9. \\ 4r-t=3. \end{cases}$

5. $\begin{cases} 3x+4r+2t=-5. \\ 2x-3r-4t=-10. \\ 4x+2r+3t=-21. \end{cases}$

6. $\begin{cases} 6A-2B-3C=3. \\ 2A+3B+5C=0. \\ 8A-5B-6C=1. \end{cases}$

7. $\begin{cases} \dfrac{2}{x}-\dfrac{1}{y}+\dfrac{1}{z}=5. \\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=9. \\ \dfrac{3}{x}-\dfrac{2}{z}=-2. \end{cases}$

8. $\begin{cases} \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{3}. \\ \dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{z}=-\dfrac{5}{3}. \\ \dfrac{1}{y}-\dfrac{1}{z}-\dfrac{1}{x}=\dfrac{25}{3}. \end{cases}$

9. $\begin{cases} x+y+2z=a. \\ 2x-y+z=b. \\ x-2y+z=c. \end{cases}$

10. $\begin{cases} \dfrac{1}{x}-\dfrac{1}{y}-\dfrac{1}{z}=a. \\ \dfrac{1}{y}-\dfrac{1}{z}-\dfrac{1}{x}=b. \\ \dfrac{1}{z}-\dfrac{1}{x}-\dfrac{1}{y}=c. \end{cases}$

11. The angle A of a triangle exceeds the angle B by 20°, and the angle C exceeds the angle A by 20°. Find the three angles of the triangle. (See § 13.)

12. The perimeter of a triangle is 175 inches. The side a is 20 inches less than twice the side b; and the side b exceeds the side c by 5 inches. Find the sides of the triangle.

13. The sum of the sides a and b of a triangle is 147 inches; the sum of the sides b and c is 135 inches; and the perimeter is 219 inches. Find the three sides of the triangle.

14. The total area of the three largest oceans is 134 million square miles. The area of the Pacific Ocean exceeds twice the area of the Atlantic Ocean by 1 million square miles; and the area of the Atlantic Ocean exceeds the area of the Indian Ocean by 7 million square miles. Find the area of each of the oceans.

15. A recipe for a fondant for candy calls for a total of three and five sixths cups of sugar, water, and glucose; twice the total amount of water and glucose exceeds the amount of sugar by one sixth cup; and three times the amount of glucose and twice the amount of sugar make six cups. How many cups of each ingredient is required for the fondant?

16. The sum of the three digits of a number is 13. If the number, decreased by 8, be divided by the sum of its units and tens' digits, the quotient is 25; and if 99 be added to the number, the digits will be reversed. Find the number.

Solution: 1. Let $x =$ the hundreds' digit,
$y =$ the tens' digit,
and $z =$ the units' digit.

2. Then $100x + 10y + z =$ the number,

3. and $100z + 10y + x =$ the number with its digits reversed.

4. By the conditions of the problem,

$$x + y + z = 13,$$

$$\frac{100x + 10y + z - 8}{y + z} = 25,$$

and $$100x + 10y + z + 99 = 100z + 10y + x.$$

17
8
126

5. Solving these questions, $x = 2$, $y = 8$, $z = 3$.
Therefore the number is 283.

17. The sum of the three digits of a number is 23; and the digit in the tens' place exceeds that in the units' place by 3. If 198 be subtracted from the number, the digits will be reversed. Find the number.

18. A and B can do a piece of work in 10 days, A and C in 12 days, and B and C in 20 days. In how many days can each of them alone do it?

XIV. SQUARE ROOT AND QUADRATIC SURDS

175. Square Root by Inspection. The square root of a perfect square monomial (§ 90), and of a perfect square trinomial (§ 96) have been found by inspection. Review these paragraphs.

176. Two Square Roots are obtained for each number. They are of equal absolute value, but have opposite signs; they may be written together, by means of the *double sign*, ±.

EXAMPLE 1. $\sqrt{9a^4b^2} = \pm 3a^2b$; since $(+3a^2b)^2 = 9a^4b^2$;
and $(-3a^2b)^2 = 9a^4b^2$.

EXAMPLE 2. $\sqrt{9a^2 - 12ax + 4x^2} = \pm(3a - 2x)$;
since $\{+(3a-2x)\}^2 = +(3a-2x)^2 = 9a^2 - 12ax + 4x^2$,
and $\{-(3a-2x)\}^2 = +(3a-2x)^2 = 9a^2 - 12ax + 4x^2$.

177. The square root of a large number may sometimes be obtained by inspection by factoring the number.

EXAMPLE. $\sqrt{1764a^4} = \sqrt{4 \cdot 441a^4} = \pm 2 \cdot 21a^2 = \pm 42a^2$.

EXERCISE 109

Find the square roots of:

1. $49m^4n^6$

2. $64x^{12}y^6z^4$.

3. $144a^4b^8c^2$.

4. $225r^6s^8t^{10}$.

5. $\dfrac{16a^{10}}{25b^4c^6}$.

6. $\dfrac{4a^4}{169b^6}$.

7. $\dfrac{196m^8}{256n^4}$.

8. $\dfrac{169x^4y^2}{81z^{12}}$.

9. $\dfrac{121c^4d^8}{144x^2y^2}$.

10. $\dfrac{225s^6t^4}{400r^{14}}$.

11. When is a trinomial a perfect square?

Find the value of:

12. $\sqrt{a^4 - 6a^2b^2 + 9b^4}$.

13. $\sqrt{m^6 - 10m^3n + 25n^2}$

14. $\sqrt{169a^2 - 26ar^3 + r^6}$.

15. $\sqrt{4x^2 - 20xy^4 + 25y^8}$.

16. $\sqrt{3136}$. **18.** $\sqrt{4225\,a^2b^2}$. **20.** $\sqrt{5184\,r^4s^2}$.

17. $\sqrt{2916\,x^4y^2}$. **19.** $\sqrt{5625\,m^4n^6}$. **21.** $\sqrt{11025\,x^2y^4z^2}$.

178. Square Root found by Long Division. If it is not possible to factor readily the number under the radical sign, the square root, if there is one, may be found by a process like long division.

EXAMPLE 1. Find the square roots of $a^2 + 2\,ab + b^2$.

SOLUTION: 1. $\sqrt{a^2} = a$. Place a in the root.
2. Square a; subtract.
3. $2 \times a = 2\,a$. Trial divisor.
$2\,ab \div 2\,a = b$. Add b to the trial divisor and to the root. Complete divisor.
4. $b \times (2\,a + b)$; subtract.

$$\begin{array}{r|l}
 & a + b \\
\hline
 & a^2 + 2\,ab + b^2 \\
 & a^2 \\
\cline{2-2}
2\,a & \quad + 2\,ab + b^2 \\
+\,b & \\
\cline{1-1}
2\,a + b & \\
 & \quad + 2\,ab + b^2 \\
\cline{2-2}
\end{array}$$

The square roots are: $+(a + b)$ and $-(a + b)$.

EXPLANATION: 1. Find the square root of the first term, obtaining a, the first term of the root; place it in the root.

2. Square the first term of the root and subtract it from the given number, obtaining the first remainder, $2\,ab + b^2$.

3. Double the first term of the root, obtaining $2\,a$, the trial divisor. Divide the first term of the remainder by $2\,a$, obtaining b, the second term of the root. Add b to the root and to the trial divisor; the complete divisor is $2\,a + b$.

4. Multiply the complete divisor by b and subtract.

Step 3 is suggested by the process of squaring a binomial. When squaring a binomial, the middle term is obtained by taking twice the product of the first and second terms; this is equivalent to taking twice the first term and multiplying by the second Reversing the process, the second term, b, will be found, if $2\,ab$ is divided by $2\,a$. After a^2 is subtracted from $a^2 + 2\,ab + b^2$, the remainder $2\,ab + b^2$ equals $b(2\,a + b)$. This suggests adding b to the trial divisor and multiplying the sum by b.

EXAMPLE 2. Find the square root of

$$20\,x^3 - 70\,x + 4\,x^4 + 49 - 3\,x^3.$$

Solution: 1. Arrange it in the descending powers of x:

2. $\sqrt{4x^4} = 2x^2$.

3. $(2x^2)^2 = 4x^4$. Subtract.

4. $2 \times (2x^2) = 4x^2$.

$20x^3 \div 4x^2 = 5x$.

5. $5x(4x^2 + 5x)$.

6. $2 \times (2x^2 + 5x)$.

7. $-28x^2 \div 4x^2 = -7$.

8. $-7(4x^2 + 10x - 7)$.

$$\begin{array}{r|l}
 & 2x^3 + 5x - 7 \\ \hline
 & 4x^4 + 20x^3 - 3x^2 - 70x + 49 \\
 & 4x^4 \\ \hline
4x^2 & 20x^3 - 3x^2 - 70x + 49 \\
+5x & \\ \hline
4x^2 + 5x & 20x^3 + 25x^2 \\ \hline
4x^2 + 10x & -28x^2 - 70x + 49 \\
-7 & \\ \hline
4x^2 + 10x - 7 & -28x^2 - 70x + 49 \\ \hline
\end{array}$$

The square roots are: $+(2x^2 + 5x - 7)$ and $-(2x^2 + 5x - 7)$.

Rule. — To find the square root of an algebraic expression:

1. Arrange it according to ascending or descending powers of some letter.

2. Write the positive square root of the first term of the given expression as the first term of the root. Square it and subtract the result from the given expression.

3. Double the root already found, for the trial divisor. Divide the first term of the remainder by the first term of the trial divisor. Add the quotient to the root and also to the trial divisor, obtaining the complete divisor.

4. Multiply the complete divisor by the new term in the square root, subtract the product from the remainder obtained in step 2.

5. Continue in this manner: (*a*) double the root already found for a new trial divisor; (*b*) divide the first term of the remainder by the first term of this product for the new term of the root; (*c*) add the new term of the root to the trial divisor, obtaining the complete divisor; (*d*) multiply the complete divisor by the new term of the root; (*e*) subtract.

EXERCISE 110

Find the square roots of the following:

1. $9x^2 + 24xy + 16y^2$.
2. $25m^2 + 30mn + 9n^2$.
3. $36a^2 - 12ab + b^2$.
4. $4x^4 + 4x^3 + 5x^2 + 2x + 1$.
5. $1 - 6a + 11a^2 - 6a^3 + a^4$.
6. $9x^4 - 24x^3 + 4x^2 + 16x + 4$.
7. $49a^2 - 30a^3 + 16 + 9a^4 - 40a$.
8. $25x^4 - 20x^3y - 26x^2y^2 + 12xy^3 + 9y^4$.

9. $9a^4+1-4a^3+4a^6-6a^2+12a^5$.

10. $16m^4+8m^3x^2-23m^2x^4-6mx^6+9x^8$.

11. $1-2x+3x^2-4x^3+3x^4-2x^5+x^6$.

12. $x^6-4x^4a^2+10x^3a^3+4x^2a^4-20xa^5+25a^6$.

13. $9x^2+25y^2+16z^2+30xy-24xz-40yz$.

14. $a^2+b^2+c^2-2ab-2ac+2bc$. 2.236

15. $20ab^3+9a^4-26a^2b^2+25b^4-12a^3b$.

16 $20x^3-70x+4x^4+49-3x^2$.

17. $49m^4-14m^3n-55m^2n^2+8mn^3+16n^4$.

18 $m^2+8m+12-\frac{16}{m}+\frac{4}{m^2}$.

19. $x^2-xy-\frac{11y^2}{4}+\frac{3y^3}{2x}+\frac{9y^4}{4x^2}$.

20. $\frac{16}{9}+\frac{8x}{3a}-\frac{13x^2}{3a^2}-\frac{4x^3}{a^3}+\frac{4x^4}{a^4}$.

179. Square Root of an Arithmetical Number. The square root of 100 is 10; of 10,000 is 100; etc. Hence the square root of a number between 1 and 100 is between 1 and 10; the square root of a number between 100 and 10,000 is between 10 and 100; etc.

That is, the integral part of the square root of a number of one or two figures contains *one* figure; of a number of three or four figures, contains *two* figures; and so on.

Hence, if the given number is divided into groups of two figures each, beginning with the units figure, for each group in the number there will be one figure in the square root. The groups are called **Periods**.

Thus, 2345 becomes 23 45; it has two periods and its square root has two figures, a tens' and a units' figure.

34038 becomes 3 40 38; it has three periods and its square root has three figures. A number having an odd number of figures will always have only one figure in its left-hand period, as in this case.

A decimal number is divided in the same manner, starting from the decimal point in both directions.

Thus, 3257.846 becomes 32 57.84 60. The last decimal period is always completed by annexing a zero. This number has two figures before the decimal point and two after it, in its square root.

180. The first figure of the square root of a number is found by inspection; the remaining figures are found in the same manner as the square root of a polynomial.

Example 1. Find the square root of 4624.

Solution. 1. Divide 4624 into periods; this gives 46 24. There are in the square root a tens' and a units' figure.

2. The tens' figure must be 6; 7 is too large for $70^2 = 4900$, which is more than 4624.

3. The rest of the square root is found as follows:

3600 is the largest square less than 4600.		60 + 8
$\sqrt{3600} = 60$; place 60 in the root.		46 24
Square 60 and subtract.		36 00
Double 60. Trial divisor.	120	10 24
$102 \div 12 = 8^+$. Place 8 in root and add to trial divisor.	8	
Complete divisor	128	10 24
Multiply complete divisor by 8.		

The square roots are + 68 and − 68.

It is customary to abbreviate the solution by omitting the zeros as in the following example.

Example 2. Find the square root of 552.25.

		23.5
Solution. The largest square less than 5 is 4; $\sqrt{4} = 2$.		5 52.25
Place 2 in the root.		4
$2 \times 2 = 4$; annex 0. Trial divisor.	40	1 52
$15 \div 4 = 3^+$; add 3 to the trial divisor.	3	
Complete divisor. Multiply by 3.	43	1 29
$2 \times 23 = 46$; annex 0. Trial divisor.	460	23 25
$230 \div 46 = 5^+$. Add 5 to the trial divisor.	5	
Complete divisor. Multiply by 5.	465	23 25

The square roots are + 23.5 and − 23.5.

Rule. — To find the square root of an arithmetical number:

1. Separate the number into periods (§ 179).

2. Find the greatest square number in the left-hand period; write its positive square root as the first figure of the root; subtract the square of the first root figure from the left-hand period, and to the result annex the next period.

3. Form the trial divisor by doubling the root already found and annexing zero.

4. Divide the remainder by the trial divisor, omitting the last figure of each. Annex the quotient to the root already found; add it to the trial divisor for the complete divisor.

5. Multiply the complete divisor by the root figure last obtained and subtract the product from the remainder.

6. If other periods remain, proceed as before, repeating steps 3, 4, and 5 until there is no remainder or until the desired number of decimal places has been obtained for the root.

NOTE 1. It sometimes happens that, on multiplying a complete divisor by the figure of the root last obtained, the product is greater than the remainder. In such cases, the figure of the root last obtained is too great, and the next smaller integer must be substituted for it.

NOTE 2. If any figure of the root is 0, annex 0 to the trial divisor and annex to the remainder the next period.

EXAMPLE 3. Find the square root of 4944.9024.

SOLUTION:

$$\begin{array}{r|l} & 70.32 \\ \hline & 49\ 44.90\ 24 \\ & 49 \\ \hline 1400 & 44\ 90 \\ 3 & \\ \hline 1403 & 42\ 09 \\ \hline 14060 & 2\ 81\ 24 \\ 2 & \\ \hline 14062 & 2\ 81\ 24 \\ \hline \end{array}$$

The square roots are + 70.32 and − 70.32.

The first trial divisor is 140. Since this is greater than 44, the first remainder, annex 0 to the root, obtaining 70.

The second trial divisor is 1400; ($2 \times 70 = 140$; annex 0, 1400). Bring down the next period 90, getting for the second remainder 4490. Divide 44 by 14 gives 3^+; annex 3 to the root and add 3 to 1400, etc.

EXERCISE 111

Find the square roots of:

1. 1521.
2. 4489.
3. 5625.
4. 8836.
5. 23409.
6. 54756.
7. 173889.
8. 42025.
9. 46225.
10. 9.8596.
11. 11.9716.
12. 17.8929.

181 The **Approximate Square Root** of a number which is not a perfect square is often desired. Obtain usually the first three figures following the decimal point.

Example. Find the approximate square root of 2.

Solution:

```
               1.414
             |2.00 00 00
             |1
       2 0   |1 00
         4   |
       2 4   |  96
        2 8 0 |4 00
            1 |
        2 8 1 |2 81
      2 8 2 0 |1 19 00
            4 |
      2 8 2 4 |1 12 96
                   7 04
```

The square roots are + 1.414+ and − 1.414+.

Note. In order to obtain the desired number of decimal places, annex zeros until there are three periods.

EXERCISE 112

Find the approximate square roots of:

1. 3.
2. 5.
3. 6.
4. 7.
5. 10.
6. 11.
7. 13.
8. 14.
9. 15.
10. 17.
11. 19.
12. 21.

182. Table of Square Roots. In the remainder of the course, it will be necessary to use frequently the square roots of some numbers. Retain some of the square roots as they are found, either in a notebook or in some other convenient place. Make a list of the numbers from 1 to 50, and write their square roots beside them, thus:

NUMBER	SQUARE ROOT
1	1.000
2	1.414
3	1.732

After working Exercise 112, twelve of the numbers of this table may be tabulated. These roots may be used to obtain the square roots of other numbers.

EXAMPLE 1. Find the square root of 8.

SOLUTION: $\sqrt{8} = \sqrt{4 \times 2} = 2 \times \sqrt{2} = 2 \times (1.414^{+}) = 2.828^{+}$.

EXAMPLE 2. Find the square root of 12.

SOLUTION: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} = 2 \times (1.732^{+}) = 3.464^{+}$.

EXERCISE 113

1. Find the following square roots to three decimals:

(a) $\sqrt{18}$. (b) $\sqrt{20}$. (c) $\sqrt{24}$. (d) $\sqrt{27}$. (e) $\sqrt{28}$.

2. Complete your table of square roots up to 50. Get as many roots as possible by inspection (§ 175); get as many of the remaining roots as possible as in Example 1. Find the others by the long division method (§ 180).

183 The square root of a fraction which is not a perfect square may be found as follows:

$$\sqrt{\frac{3}{2}} = \sqrt{\frac{3 \times 2}{2 \times 2}} = \sqrt{\frac{6}{4}} = \frac{\sqrt{6}}{\sqrt{4}} = \pm\frac{\sqrt{6}}{2}.$$

$$\pm\frac{\sqrt{6}}{2} = \pm\frac{2.449^{+}}{2} = \pm 1.224^{+}.$$

Rule. — To find the square root of a fraction:

1. Change the fraction into an equivalent fraction with a perfect square denominator.

2. The square root of the new fraction equals the square root of its numerator divided by the square root of its denominator.

3. If desired, express the result of step 2 in simplest decimal form, prefixing the double sign, ±.

EXAMPLE. Find the approximate square root of $\frac{3}{8}$.

SOLUTION: 1. The smallest square number into which 8 can be changed is 16; multiply both terms of the fraction by 2.

2. $\sqrt{\frac{3}{8}}=\sqrt{\frac{2\times 3}{2\times 8}}=\sqrt{\frac{6}{16}}=\pm\frac{\sqrt{6}}{4}=\pm\frac{2.449^{+}}{4}=\pm .612^{+}.$

EXERCISE 114

Find the approximate square roots of:

1. $\frac{3}{4}$.
2. $\frac{5}{9}$.
3. $\frac{7}{16}$.
4. $\frac{1}{3}$.
5. $\frac{1}{2}$.
6. $\frac{2}{5}$.
7. $\frac{3}{7}$.
8. $\frac{5}{8}$.
9. $\frac{5}{6}$.
10. $\frac{4}{7}$.
11. $\frac{9}{5}$.
12. $\frac{3}{10}$.
13. $\frac{7}{11}$.
14. $\frac{5}{18}$.
15. $\frac{2}{27}$.

QUADRATIC SURDS

184. The indicated square root of a number which is not a perfect square is called a **Quadratic Surd**; as, $\sqrt{3}$, $\sqrt{\frac{5}{3}}$, $\sqrt{x}$, $\sqrt{\frac{x}{y}+1}$.

185. Surds should be simplified as in the following examples:

(a) $\sqrt{24}=\sqrt{4\cdot 6}=2\cdot\sqrt{6}$; (b) $\sqrt{\frac{9}{8}}=\sqrt{\frac{9\cdot 2}{16}}=\frac{3\sqrt{2}}{4}$.

Thus, a quadratic surd is in its simplest form when the number under the radical sign is an integer which does not contain any perfect square factor.

While a quadratic surd has two values, one positive and one negative, it is agreed to consider only the positive root, in order to avoid ambiguity. This root is called the *principal root*.

186. Addition and Subtraction of Surds.

EXAMPLE 1. Find the sum of $\sqrt{20}$ and $\sqrt{45}$.

SOLUTION: 1. $\sqrt{20}+\sqrt{45}=\sqrt{4\cdot 5}+\sqrt{9\cdot 5}=2\sqrt{5}+3\sqrt{5}=5\sqrt{5}.$

This solution assumes that surds may be added like other numbers. The coefficients of $\sqrt{5}$ are 2 and 3; the sum is found by multiplying $\sqrt{5}$ by the sum of its coefficients (§ 34).

The advantage in adding surds in this way is that fewer square roots need be obtained. Thus, the sum of $\sqrt{20}$ and $\sqrt{45}$ is $5\sqrt{5}$ or $5 \times (2.236^+)$ or 11.180^+. This same result could be obtained by adding the square roots of 20 and 45.

EXAMPLE 2. Simplify $\sqrt{\frac{9}{8}} + \sqrt{\frac{1}{2}}$.

SOLUTION: 1. $\sqrt{\frac{9}{8}}+\sqrt{\frac{1}{2}}=\sqrt{\frac{18}{16}}+\sqrt{\frac{2}{4}}=\frac{3\sqrt{2}}{4}+\frac{\sqrt{2}}{2}=\frac{3\sqrt{2}}{4}+\frac{2\sqrt{2}}{4}=\frac{5\sqrt{2}}{4}$.

2. $\frac{5\sqrt{2}}{4}=\frac{5\times(1.414^+)}{4}=\frac{7.070^+}{4}=1.767^+$.

EXAMPLE 3. Simplify $\frac{2}{3} + \sqrt{\frac{1}{3}}$.

SOLUTION: 1. $\frac{2}{3}+\sqrt{\frac{1}{3}}=\frac{2}{3}+\sqrt{\frac{3}{9}}=\frac{2}{3}+\frac{\sqrt{3}}{3}=\frac{2+\sqrt{3}}{3}$.

2. $\frac{2+\sqrt{3}}{3}=\frac{2+1.732^+}{3}=\frac{3.732^+}{3}=1.244^+$.

NOTE. The results of problems involving surds are often left in the surd form as in step 1 of Examples 2 and 3. There are advantages in finding the approximate decimal value of the result.

Tues 8

EXERCISE 115

Simplify the following:

$+\sqrt{2}$ 3

1. $\sqrt{12}+\sqrt{75}$.
2. $\sqrt{98}-\sqrt{18}$.
3. $\sqrt{80}-\sqrt{20}$.
4. $3\sqrt{27}-\sqrt{48}$.
5. $\sqrt{28}+\sqrt{63}$.
6. $6\sqrt{2}-\sqrt{50}+\sqrt{18}$.
7. $\sqrt{12}+\sqrt{\frac{1}{3}}$.
8. $\sqrt{\frac{2}{5}}+\sqrt{\frac{1}{10}}$.
9. $\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{27}}$.
10. $\sqrt{\frac{3}{8}}-\sqrt{24}$.
11. $\frac{2}{3}+\sqrt{\frac{5}{9}}$.
12. $\frac{1}{2}-\sqrt{\frac{3}{4}}$.
13. $\frac{3}{5}+\sqrt{\frac{2}{25}}$.
14. $-\frac{1}{7}+\sqrt{\frac{2}{49}}$.
15. $-\frac{4}{11}+\sqrt{\frac{1}{121}}$.
16. $+\frac{2}{3}+\sqrt{\frac{4}{3}}$.
17. $-\frac{5}{6}+\sqrt{\frac{1}{6}}$.
18. $-\frac{3}{7}+\sqrt{\frac{2}{7}}$.
19. $-\frac{3}{8}+\sqrt{\frac{5}{8}}$.
20. $\frac{3}{5}-\sqrt{\frac{19}{5}}$.

XV. QUADRATIC EQUATIONS

187. A **Quadratic Equation** is an equation of the second degree (§ 161); it may have one or more unknowns.

188. A **Pure Quadratic Equation** is a quadratic equation having only one unknown, which contains only the second power of the unknown, as, $ax^2 = b$

EXAMPLE 1. An acre of ground contains 43,560 square feet. How long must the side of a square field be in order that the area of the field shall be one acre?

SOLUTION: 1. Let $s =$ the number of feet in one side.

2. Then $s^2 =$ the number of square feet in the area.

3. Then $s^2 = 43{,}560.$

Extract the square root of both members of the equation.

4. Then $s = \pm 208.7^{+}.$

Since this is a field, only the positive root has meaning; hence the side of the field must be 208.7^{+} feet.

189. A pure quadratic equation has two roots, because two square roots are obtained in extracting the square roots of the two members of the equation.

Rule. — To solve a pure quadratic equation.

1. Clear the equation of fractions, transpose, and combine terms until the equation takes the form $x^2 =$ a number.

2. Extract the square root of both members of the equation, placing the double sign, ±, before the root in the right member.

NOTE. After extracting the square roots of both members of an equation like $x^2 = a^2$, we get $\pm x = \pm a$. This gives: $+x = +a$, $+x = -a$, $-x = +a$, and $-x = -a$.

If both members of the last two equations are multiplied by -1, the equations become $+x=-a$, and $+x=+a$. These are the first two of our four equations. Thus, it is clear that, from $x^2=a^2$, we get only *two* equations, $x=+a$ and $x=-a$, or $x=\pm a$.

Example. Solve the equation $\frac{2m}{3}+\frac{3}{m}=\frac{m}{12}+\frac{12}{m}$.

Solution: 1. $\frac{2m}{3}+\frac{3}{m}=\frac{m}{12}+\frac{12}{m}$.

2. M_{12m}: $8m^2+36=m^2+144$.

3. Simplifying: $7m^2=108$.

4. D_7: $m^2=\frac{108}{7}$.

5. $\sqrt{\ }$:* $m=\pm\sqrt{\frac{108}{7}}=\pm 6\sqrt{\frac{3}{7}}=\pm\frac{6}{7}\sqrt{21}$.

6. $\sqrt{21}=4.582$: $m=\pm\frac{6}{7}\cdot(4.582)=\pm\frac{27.492}{7}=\pm 3.927$.

7. $m_1=+3.927$; $m_2=-3.927$.

"m_1" is read "m one." The numeral 1 is called in such cases a subscript. "m_2" is read "m two." These subscripts are used in this case to distinguish between the two roots of the quadratic.

Check: In cases such as this, it is better to check by going over the solution a second time. Great care must be taken, however, for it is easy to overlook an error.

Note.— Get the result in the radical form first; that is, $m=\pm\frac{6}{7}\sqrt{21}$; then it is wise, for many reasons, to get it in decimal form as finally given.

EXERCISE 116

Solve the following equations:

1. $7p^2-175=0$.
2. $5x^2-48=80-3x^2$.
3. $12c^2-140=9c^2-32$.
4. $\frac{2m^2+4}{5}-\frac{3m^2-7}{3}=\frac{11}{15}$.
5. $3(m-2)+2m(m-1)=m$.
6. $3(t+5)-t(t-1)=4t$.
7. $9a^2-7=0$.
8. $11a^2-3=1$.

---

* The symbol "$\sqrt{\ }$:" placed in the left margin will mean, "take the square root of both members of the previous equation."

Review the statement made at the top of page 187; bear it in mind whenever solving fractional equations.

9. $\frac{7}{3S^2} - \frac{11}{9S^2} = \frac{5}{6}$.

10. $\frac{2r-3}{4-r} = \frac{9+r}{3r+2}$.

11. $\frac{x^2+x+1}{x-1} - \frac{x^2-x+1}{x+1} = 6$.

12. $\frac{3t^2+2}{5} - \frac{5t^2-3}{10} = \frac{4t^2+1}{25}$.

13. $\frac{g+4}{g-4} + \frac{g-4}{g+4} = \frac{10}{3}$.

14. $\frac{r^4-3r^2+4}{3r^4+2r^2-4} = \frac{r^2-3}{3r^2+2}$.

15. $a^2 - 2cx^2 = 3b^2$. Solve for x.

SOLUTION: 1. $-2cx^2 = 3b^2 - a^2$.

2. $2cx^2 = a^2 - 3b^2$.

3. $x^2 = \frac{a^2-3b^2}{2c}$.

4. $$x = \pm\sqrt{\frac{a^2-3b^2}{2c}} = \pm\sqrt{\frac{2c(a^2-3b^2)}{4c^2}}$$
$$= \pm\frac{1}{2c}\sqrt{2a^2c - 6b^2c}.$$

Solve for x:

16. $a^2 + x^2 = m$.

17. $a + 2x^2 = c$.

18. $ax^2 = m$.

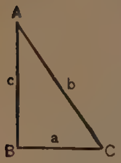
19. $2cx^2 = d^2$.

20. $\frac{1}{2}hx^2 = c$.

21. $3mx^2 - n = p$.

hee

190. **A Right Triangle** is a triangle which has a right angle for one of its angles; as triangle ABC, in which angle B is a right angle.



The side opposite the right angle is the *hypotenuse;* as, side AC. The side BC is the *base* and AB is the *altitude.*

In a right triangle, the square of the hypotenuse equals the sum of the squares of the other two sides.

Thus, $b^2 = a^2 + c^2$.

To verify this fact, draw a right triangle with BC 3 inches and AB 4 inches; measure AC. Substitute the lengths of the sides in the equation $b^2 = a^2 + c^2$.

EXERCISE 117

Carry out all results in this set to one decimal place:

1. Find the altitude of a right triangle whose base is 13 feet and whose hypotenuse is 30 feet.

SOLUTION: 1. Let $x =$ the number of feet in the altitude.
2. Then $x^2 + 13^2 = 30^2$. (why?)
3. Complete the solution.

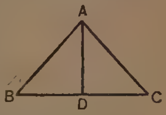
2. Find the base of a right triangle whose hypotenuse is 45 feet and whose altitude is 25 feet.

3. If the diagonal of a rectangle is 68 inches and the base of the rectangle is three times the altitude, what are the dimensions of the rectangle?

4. If the altitude of a rectangle is a and the base is 4 times the altitude, what is the length of the diagonal?

5. Solve the formula $b^2 = a^2 + c^2$: (a) for a; (b) for c.

In the isosceles triangle ABC, AD, which is perpendicular to BC, is the altitude and BC is the base. BD and DC are equal; this may be verified by measuring them.



6. If AB is 15 inches and BC is 18 inches, find AD.

7. If the equal sides of an isosceles triangle are each 30 inches, and the altitude is 18 inches, find the base.

8. If the equal sides of an isosceles triangle are each $3a$ inches and the base is $2a$ inches, find the altitude?

9. An equilateral triangle is one which has all of its sides equal. Find the altitude of an equilateral triangle whose sides are each 10 inches.

10. (a) Find the altitude of an equilateral triangle whose sides are each s inches; (b) find the area of the triangle.

11. The area of a circle is found by the formula $A = \pi r^2$, where r is the radius, and $\pi = 3\frac{1}{7}$.

Find the area of a circle whose radius is 9 inches.

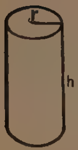
12. What is the radius of a circle whose area is an acre? (Express the radius in feet See § 188.)

Express the results of the following in simplest radical form).

13. Solve the formula $A = \pi r^2$ for r: (a) letting $\pi = 3\frac{1}{7}$; (b) without substituting the value of π.

14. The volume of a circular cylinder is given by the formula $V = \pi r^2 h$; where $r =$ the radius, h the altitude.

Find V when $r = 5$ and $h = 13$.



15. Find r when $V = 759$ and $h = 14$.

16. Solve the cylinder formula for r.

17. Solve the formula $S = 4\pi r^2$ for r.

18 Solve the formula $V = \frac{1}{3}\pi r^2 h$ for r.

19. The distance s in feet through which an object will fall in t seconds is given by the formula $s = \frac{1}{2}gt^2$, where g is 32.

Suppose that a stone is allowed to fall from a tower; how far will it fall in: (a) 3 seconds? (b) 5 seconds?

20 How long will it take a ball to fall 900 feet?

21. Washington's Monument in Washington, D.C., is 555 feet high. How long will it take a ball to fall that distance?

22. Solve the equation $s = \frac{1}{2}gt^2$ for t.

23 Solve the formula $f = \dfrac{mv^2}{r}$ for v.

COMPLETE QUADRATIC EQUATIONS

191. **A Complete Quadratic Equation** is a quadratic equation having only one unknown, which contains the first power of the unknown as well as the second power; as,

$$2x^2 - 3x - 5 = 0.$$

192. Complete quadratic equations have been **Solved by Factoring** in § 108.

EXERCISE 118

Solve by factoring:

1. $x^2 - 5x = 24.$

2. $2m^2 - m - 15 = 0.$

3. $2x^2 + 13x + 20 = 0.$

4. $12y^2 - 7y = 10.$

5. $5w^2 - 7w = 0$

6. $9x^2 - 49x = 0.$

7. $2x^2 + ax - a^2 = 0.$
(Solve for x).

8. $8x^2 + 14mx = -3m^2.$

9. $6x^2 - 11xk = 10k^2.$

10. $5x^2 + 15p^2 = 28xp.$

11. $\frac{x^2}{3} - \frac{x}{2} = \frac{35}{6}.$

12. $\frac{1}{2} - \frac{5}{6x^2} = -\frac{7}{12x}.$

13. $\frac{1}{8x^2} - \frac{13}{24x} = -\frac{1}{2}.$

14. $\frac{x}{4} + \frac{4}{x} = -\frac{29}{10}.$

15. $\frac{4-t}{1-t} = \frac{12}{3-t}.$

16. $\frac{7}{s-3} = \frac{s-2}{s-4} + \frac{1}{2}.$

17. $\frac{7}{y-3} - \frac{3}{y-4} = \frac{1}{2}.$

18. $1 + \frac{2}{p-3} - \frac{6}{p-8} = 0.$

19. $\frac{3}{m+6} - \frac{2}{m-5} = \frac{5}{4}.$

20. $\frac{r+2}{r+3} = \frac{36}{(r+3)^2} - 1.$

193. Graphical Solution of Equations With One Variable. Many facts about equations containing one variable can be discovered by the aid of graphical representation.

EXAMPLE 1. Consider the equation $3x - 12 = 0.$

The expression $3x - 12$ has a different value for each value of x:

Thus if $x = 2$, $3x - 12 = -6$; if $x = -3$, $3x - 12 = -21.$

The problem is to find the value of x for which the expression $3x - 12$ will equal zero.

GRAPHICAL SOLUTION: 1. Let $y = 3x - 12.$

2. Find values of y for some values of x:

if $x = 0$, $y = -12.$ if $x = -2$, $y = -18.$

if $x = +5$, $y = +3.$ if $x = +6$, $y = +6.$

3. Use these pairs of numbers as coördinates of points and draw the graph.

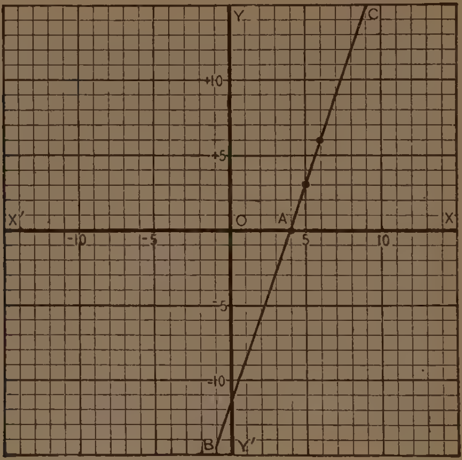


Fig. 9

4. *BC* crosses the x axis at point *A*. The coördinates of *A* are: $x = 4$, $y = 0$.

5. Hence when $x = 4$, $3x - 12 = 0$. (y is the expression $3x - 12$.)

$\therefore$ $x = 4$ is the desired solution of the equation, for we were looking for a value of x for which $3x - 12 = 0$.

Rule. — To solve graphically an equation containing one variable:

1. Simplify the equation as much as possible.

2. Transpose all terms to the left member.

3. Represent by y the expression found in step 2.

4. Find for y the values which correspond to selected values of the variable in the equation.

5. Use the pairs of values obtained in step 4 as coördinates of points; plot the points; draw the graph, making the vertical axis the y axis.

6. The graph crosses the horizontal axis at points whose ordinates are zero, and whose abscissæ are the desired roots of the equation.

Example 2. Solve the equation $x^2 - x = 6$.

Solution: 1. $x^2 - x = 6$, or $x^2 - x - 6 = 0$.

2. Let $y = x^2 - x - 6$.

3. If $x = -4$, $y = (-4)^2 - (-4) - 6 = 16 + 4 - 6 = +14$.

4\.

Similarly if $x =$	0	$+1$	$+2$	$+4$	$+5$	-1	-2	-3	-4
then $y =$	-6	-6	-4	$+6$	$+14$	-4	0	$+6$	$+14$

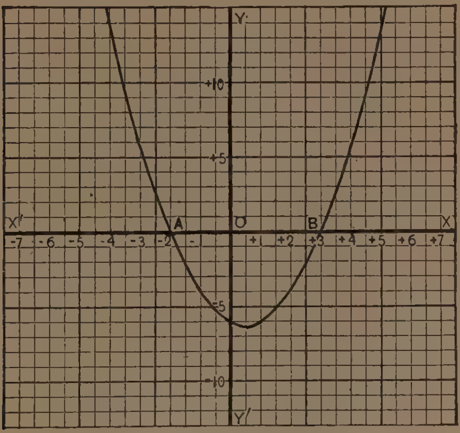


Fig. 10

5. The graph crosses the horizontal axis at the points A and B. According to the rule, the abscissæ of these points are the two roots of the equation.

At A: $x = -2$, $y = 0$; *i.e.* $x^2 - x - 6 = 0$.

At B: $x = +3$, $y = 0$; *i.e.* $x^2 - x - 6 = 0$.

Check: $x = -2$; does $(-2)^2 - (-2) - 6 = 0$? Yes.

$x = +3$; does $(+3)^2 - (+3) - 6 = 0$? Yes.

Linear

EXERCISE 119

Solve graphically the equations:

1. $x + 3 = 0$.
2. $2x = 7$.
3. $x^2 = 16$.
4. $x^2 - x = 12$.
5. $x^2 - 7x + 6 = 0$.
6. $x^2 + 6x + 5 = 0$.

194. Some quadratic equations cannot be solved readily by factoring; for example, $x^2 - 6x - 2 = 0$, since $x^2 - 6x - 2$ does not have any rational factors. The graphical solution shows that this equation has two roots.

SOLUTION: 1. $x^2 - 6x - 2 = 0$.

2. Let $y = x^2 - 6x - 2$.

When $x =$	-2	-1	0	$+1$	$+2$	$+3$	$+4$	$+6$	$+7$	$+8$
then $y =$	$+14$	$+5$	-2	-7	-10	-11	-10	-2	$+5$	$+14$

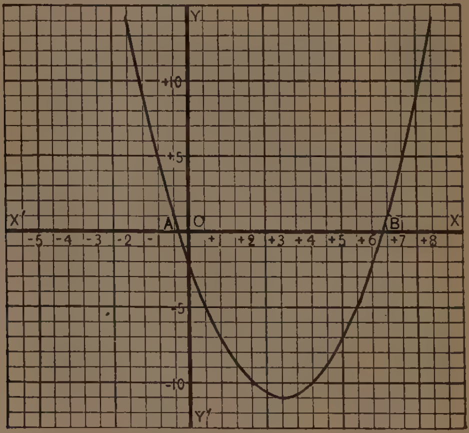


FIG. 11

3. The graph crosses the horizontal axis at the points A and B. The abscissa of A is about $-.3$: the abscissa of B is about $+6.3$.

This indicates that the roots of this equation are approximately $-.3$, and $+6.3$.

There are two methods of solving such equations which give the roots more accurately.

195 Solution by Completing the Square.

DEVELOPMENT 1. Find: (*a*) $(x-4)^2$; (*b*) $(x+5)^2$;
(*c*) $(x-\frac{2}{3})^2$; (*d*) $(y+\frac{3}{4})^2$.

2. When is a trinomial a perfect square? (See § 96.)

3. Make a perfect square trinomial of x^2-10x.

SOLUTION 1. $\frac{1}{2}$ of $10=5$; $5^2=25$; add 25
2. The perfect square is $x^2-10x+25$ or $(x-5)^2$.

4. Make perfect square trinomials of the following:

(*a*) x^2-12x; (*b*) y^2-14y; (*c*) z^2-20z.

5. Solve the equation $x^2-12x+20=0$

SOLUTION: 1. S_{20}: $x^2-12x=-20$.

2. Make the left member a perfect square by adding 36; therefore add 36 to both members: (§ 41).

A_{36}: $x^2-12x+36=36-20$.
or $(x-6)^2=16$.
3. $\sqrt{\ }$: $x-6=\pm 4$.
4 $\therefore x-6=+4$, or $x=6+4=10$, one root,
and $x-6=-4$, or $x=6-4=2$, another root.

CHECK: $x=10$; does $(10)^2-12(10)+20=0$? Yes.
$x=2$; does $(2)^2-12(2)+20=0$? Yes.

6. Solve the equation $x^2-3x-5=0$.

SOLUTION: 1. $x^2-3x-5=0$.
2. A_5: $x^2-3x=+5$.
3. $\frac{1}{2}(-3)=-\frac{3}{2}$; $(-\frac{3}{2})^2=+\frac{9}{4}$; add $\frac{9}{4}$ to both members.
4. $A_{\frac{9}{4}}$: $x^2-3x+\frac{9}{4}=5+\frac{9}{4}=\frac{29}{4}$.
5. $\sqrt{\ }$: $x-\frac{3}{2}=\pm\sqrt{\frac{29}{4}}=\pm\frac{1}{2}\sqrt{29}$.
6. $\therefore x=\frac{3}{2}\pm\frac{1}{2}\sqrt{29}=\frac{3\pm\sqrt{29}}{2}$.

7. Radical results, $x_1 = \frac{3+\sqrt{29}}{2}$ and $x_2 = \frac{3-\sqrt{29}}{2}$.

8. Decimal results, $x_1 = \frac{3+5.385}{2}$ and $x_2 = \frac{3-5.385}{2}$.

$$x_1 = \frac{8.385}{2} = 4.192^{+} \qquad x_2 = \frac{-2.385}{2} = -1.192^{+}$$

CHECK: To check the solution by substituting the roots in either their decimal or their radical form is a long process, with many opportunities for errors. Persons skillful in algebra check by going over the solution carefully.

A quick check, the reason for which will be learned later in algebra, is to find the algebraic sum of the roots; this result should equal the negative of the algebraic coefficient of x in the equation in which the coefficient of x^2 is 1.

Here: $+4.192^{+}$ The coefficient of x^2 is 1. The coefficient of x
-1.192^{+} is -3. This equals the negative of the algebraic
Sum, $+3$ sum of the roots.

If the coefficient of x^2 is not 1, first imagine the equation divided by that coefficient, and *then* select the coefficient of x.

Rule. — To solve a quadratic equation by completing the square:

1. Simplify the equation: transpose all terms containing the unknown number to the left member, and all other terms to the right member so that the equation takes the form

$$ax^2 + bx = c.$$

2. If the coefficient of x^2 is not 1, divide both members of the equation by it, so that the equation takes the form

$$x^2 + px = q.$$

3. Find one half of the coefficient of x: square the result; add the square to both members of the equation obtained in step 2. This makes the left member a perfect square.

4. Write the left member as the square of a binomial; express the right member in its simplest form.

5. Take the square root of both members, writing the double sign, ±, before the square root in the right member.

6. Set the left square root equal to the + root in the right member of the equation in step 5. Solve for the unknown. This gives one root.

7. Repeat the process, using the − root in step 5. This gives the second root of the equation.

8. Express the roots first in simplest radical form, and then, if desired, in simplest decimal form.

EXERCISE 120

Solve by completing the square:

1. $x^2 - 4x - 5 = 0.$
2. $x^2 + 8x - 33 = 0.$
3. $x^2 - 6x - 27 = 0.$
4. $x^2 + 10x + 24 = 0.$
5. $x^2 - 2x - 15 = 0.$
6. $y^2 - 4y = 4.$
7. $a^2 + 6a = 1.$
8. $m^2 - 2m = 1.$
9. $t^2 - 8t = 4.$
10. $z^2 - 10z = 5.$
11. $x^2 + 5x = -4.$
12. $s^2 + 10 = 7s.$
13. $w^2 - 3w = -2.$
14. $m^2 + m = 30.$
15. $r^2 - 13r + 30 = 0.$
16. $z^2 - 6 = 5z.$
17. $g^2 + 9g = 11.$
18. $y^2 - 15y + 16 = 0.$

19. Solve the equation $x^2 - \frac{2}{3}x = 1.$

SOLUTION: 1. $\frac{1}{2}$ of $\frac{2}{3} = \frac{1}{3}.$ $(\frac{1}{3})^2 = \frac{1}{9}.$

2. $A_{\frac{1}{9}}$: $x^2 - \frac{2}{3}x + \frac{1}{9} = 1 + \frac{1}{9}.$

3. $(x - \frac{1}{3})^2 = (\frac{10}{9}).$

4. $x - \frac{1}{3} = \pm\sqrt{\frac{10}{9}} = \pm\frac{1}{3}\sqrt{10}.$

5. $x - \frac{1}{3} = +\frac{1}{3}\sqrt{10}.$ $x - \frac{1}{3} = -\frac{1}{3}\sqrt{10}.$

$$x = \frac{1}{3} + \frac{1}{3}\sqrt{10} \qquad x = \frac{1}{3} - \frac{1}{3}\sqrt{10}$$

$$= \frac{1 + \sqrt{10}}{3} \qquad = \frac{1 - \sqrt{10}}{3}$$

$$= \frac{1 + 3.162}{3} \qquad = \frac{1 - 3.162}{3}$$

$$= \frac{4.162}{3} = +1.387^{+}. \qquad = \frac{-2.162}{3} = -.720^{+}.$$

CHECK: $+1.387$
$-.720$
Sum. $+.667$

Coefficient of $x = -\frac{2}{3} = -.6666.$
$= -.667.$
$+.667 = -(-.667).$

20. $x^2 - \frac{4}{3}x = \frac{4}{3}.$
21. $y^2 - \frac{6}{5}y = \frac{8}{5}.$
22. $z^2 + \frac{6}{7}z = 1.$
23. $a^2 - \frac{1}{3}a = \frac{2}{3}.$
24. $m^2 - \frac{3}{5}m = \frac{2}{5}.$
25. $t^2 - \frac{5}{4}t = 7.$

26. Solve the equation $3x^2 - 2x - 1 = 0.$

SOLUTION: 1. D_3: $x^2 - \frac{2}{3}x - \frac{1}{3} = 0.$

$\frac{2x}{3}$

2. $A_{\frac{1}{3}}$: $x^2 - \frac{2}{3}x = \frac{1}{3}.$

3. Complete the solution as in Examples 19–25.

27. $3x^2 - 2x - 5 = 0.$
28. $5r^2 + 2r - 3 = 0.$
29. $4t^2 - 8t + 3 = 0.$
30. $3x^2 - 4x - 7 = 0.$
31. $2a^2 - 3a - 9 = 0.$
32. $5p^2 + 3p = 1$
33. $6w^2 - 5w = 10.$
34. $\frac{3}{5a^2} - \frac{2}{a} - 5 = 0$
35. $\frac{1}{2p} + 2 - 2p = 0.$
36. $\frac{m}{3} - \frac{2}{m} + 4 = 0.$
37. $\frac{7}{r^2} + \frac{2}{r} - 5 = 0.$
38. $\frac{t}{t-3} - \frac{3t}{4} = \frac{5}{2}.$
39. $\frac{x^2}{x-4} + \frac{1}{5} = 0.$
40. $\frac{3w}{w-7} + \frac{2}{w-4} = 0.$
41. $\frac{1}{a-5} - \frac{2}{a-2} = \frac{3}{5}.$
42. $\frac{2}{c-2} - \frac{1}{c+2} = -\frac{15}{8}.$
43. $\frac{3}{s-3} - \frac{2}{s-2} = 1.$
44. $\frac{3}{2y-7} + \frac{1}{15} = \frac{2}{y-3}.$
45. $\frac{2}{w^2-9} - \frac{3w}{w+3} = \frac{1}{w-3}.$

196. Solution of Literal Quadratic Equations.

EXAMPLE. Solve the equation $ax^2 - 3bx - c = 0.$

SOLUTION: 1. $ax^2 - 3bx - c = 0.$

2. D_a: $$x^2 - \frac{3b}{a}x - \frac{c}{a} = 0.$$

3. $A_{\frac{c}{a}}$: $$x^2 - \frac{3b}{a}x = \frac{c}{a}.$$

4. The coefficient of x is $\left(\frac{-3b}{a}\right)$; one half of it is $\left(\frac{-3b}{2a}\right)$.

The square of $\left(\frac{-3b}{2a}\right)$ is $\left(\frac{9b^2}{4a^2}\right)$. Add this to both members of equation 3.

5. $$x^2 - \frac{3b}{a}x + \frac{9b^2}{4a^2} = \frac{9b^2}{4a^2} + \frac{c}{a}.$$

$$\therefore \left(x - \frac{3b}{2a}\right)^2 = \frac{9b^2 + 4ac}{4a^2}.$$

6. $$x - \frac{3b}{2a} = \pm\frac{1}{2a}\sqrt{9b^2 + 4ac}.$$

7. $$x = \frac{+3b \pm \sqrt{9b^2 + 4ac}}{2a}.$$

8. $$\therefore x_1 = \frac{+3b + \sqrt{9b^2 + 4ac}}{2a};\quad x_2 = \frac{+3b - \sqrt{9b^2 + 4ac}}{2a}.$$

Check: $x_1 + x_2 = \frac{+6b}{2a} = +\frac{3b}{a}$. Since this is the negative of the coefficient of x in step 2, the roots are correct.

EXERCISE 121

Solve the following equations for x:

1. $x^2 + 2ax - 3 = 0.$
2. $x^2 + 2ax + b = 0.$
3. $x^2 + 4x - c = 0.$
4. $x^2 + 3x + m = 0.$
5. $2x^2 + 3x - n = 0.$
6. $2x^2 + 4ax + b = 0.$
7. $x^2 + 3ax - 4t = 0.$
8. $x^2 + 2mx = 2m + 1.$
9. $x^2 - 4ax = 9b^2 - 4a^2.$
10. $ax^2 + 2x + 1 = 0.$
11. $rx^2 + 4tx - 5 = 0.$
12. $cx^2 + 2dx + p = 0.$
13. $x^2 + mx + n = 0.$
14. $ax^2 + bx + c = 0.$

197. Solution of Quadratic Equations by a Formula. All quadratic equations having one unknown may be put in the form $$ax^2 + bx + c = 0.$$

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This equation may be solved like the equations in Exercise 121. The roots will be found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

This result may be used as a *formula* for solving any quadratic equation *of the form* $ax^2 + bx + c$.

Example 1. Solve the equation $2x^2 - 3x - 5 = 0$

Solution: 1. Comparing the equation with $ax^2 + bx + c = 0$:

$$a = 2,\quad b = -3,\quad c = -5$$

2. Substitute these values in the formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Then

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$

$$= \frac{+3 \pm \sqrt{9 + 40}}{4}$$

$$= \frac{3 \pm \sqrt{49}}{4} = \frac{3 \pm 7}{4}.$$

4.

$$\therefore x_1 = \frac{3 + 7}{4} = \frac{10}{4} = \frac{5}{2};\quad x_2 = \frac{3 - 7}{4} = \frac{-4}{4} = -1.$$

Check: $x_1 = \frac{5}{2}$; does $2(\frac{5}{2})^2 - 3(\frac{5}{2}) - 5 = 0$?

does $2 \cdot \frac{25}{4} - \frac{15}{2} - 5 = 0$?

does $\frac{25}{2} - \frac{15}{2} - 5 = 0$? Yes.

$x_2 = -1$; does $2(-1)^2 - 3(-1) - 5 = 0$?

does $2 + 3 - 5 = 0$? Yes.

Example 2. Solve the equation $2x^2 - 3x - 3 = 0$.

Solution: 1. $a = 2,\ b = -3,\ c = -3$.

2. Substituting in the formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$:

$$x = \frac{3 \pm \sqrt{9 + 24}}{4} = \frac{3 \pm \sqrt{33}}{4} = \frac{3 \pm 5.744^+}{4}.$$

$$\therefore x_1 = \frac{8.744^+}{4} = 2.186^+;\quad x_2 = \frac{-2.744^+}{4} = -.686^+$$

$ax^2 + bx + c = 0$

Check: $\begin{array}{r} 2.186+ \\ -\ .686+ \\ \hline 1.500 \end{array}$ The coefficient of x is $-\frac{3}{2}$, when the coefficient of $x^2 = 1$; $1.5 = -(-\frac{3}{2})$.

(For this method of checking, see § 195.)

EXERCISE 122

Solve the equations:

1. $4r^2 - 7r + 3 = 0$.
2. $6t^2 + 13t + 5 = 0$.
3. $y^2 = 6y + 72$.
4. $a^2 - 7a - 30 = 0$.
5. $3x^2 - 2x - 33 = 0$.
6. $3m^2 + 5m + 1 = 0$.
7. $5s^2 = 5s - 1$.
8. $6w^2 - 11w + 2 = 0$.
9. $2x^2 - 3x - 1 = 0$.
10. $x^2 + x - 1 = 0$.
11. $y^2 - 4y + 2 = 0$.
12. $3w^2 - 7w + 2 = 0$.
13. $5c^2 + 8c = 4$.
14. $9x^2 + 16x + 3 = 0$.
15. $15d^2 - 22d - 5 = 0$.
16. $\frac{x^2}{3} - \frac{x}{2} - \frac{3}{2} = 0$.
17. $\frac{4}{x} - x - \frac{1}{3} = 0$.
18. $\frac{3}{5-t} - 2t = 0$.
19. $\frac{2}{d-1} - \frac{3}{d} = 5$.

198. Summary of Methods of Solving a Quadratic. Four methods of solving a quadratic equation have been given: the graphical, by factoring, by completing the square, and by the formula. The first is useful mainly as a means of illustration; the third is useful mainly in solving the general quadratic $ax^2 + bx + c = 0$, and, thus, in deriving the formula.

Historical Note. Greek mathematicians as early as Euclid were able to solve certain quadratics by a geometric method, about which the student may learn when he studies plane geometry. Heron of Alexandria, about 110 B.C., proposed a problem which leads to a quadratic. His solution is not given, but his result would indicate that he probably solved the equation by a rule which might be obtained from the quadratic by completing its square in a certain manner. Diophantus, 275 A.D., gave many problems which lead to quadratic equations. The rules by which he solved his equations appear to have been derived by completing the square. He considered three separate kinds of quadratics. He gave only one root for a quadratic, even when the equation had two roots.

The Hindu mathematicians, knowing about negative numbers, considered one general quadratic. Cridharra gave a rule much like our formula. The Hindus knew that a quadratic has two roots, but they usually rejected any negative roots.

The Arabians went back to the practice of Diophantus in considering three or more kinds of quadratics. Mohammed Ben Musa, 820 A.D., had five kinds. He admitted two roots when both were positive. Alkarchi gave a purely algebraic solution of a quadratic by completing the square, and refers to this method as being a diophantic method.

In Europe, mathematicians followed the practice of the Arabians, and by the time of Widmann, 1489, had twenty four special forms of equations. These were solved by rules which were learned and used in a mechanical manner. Stifel, 1486–1567, finally brought the study of quadratics back to the point that had been reached by the Hindus one thousand years before. He gave only three normal forms for the quadratic; he allowed double roots when they were both positive. Stevin, 1548–1620, went still farther. He gave only one normal form for the general quadratic, as do we; he solved this in both a geometric and an algebraic manner, giving the method of completing the square. He allowed negative roots.

EXERCISE 123

Miscellaneous Examples

Solve the following equations by any of the preceding methods. As a rule, solve by factoring if possible, otherwise by the formula.

1. $(3x+2)(2x+3)=(x-3)(2x-4)$.

2. [illegible]$(y-1)^2-4(y-2)^2=44$.

3. $\dfrac{30}{m}-\dfrac{30}{m+1}=1$.

4. $\dfrac{3}{a-6}-\dfrac{2}{a-5}=1$.

5. $\dfrac{5}{x+4}-\dfrac{3}{x-2}=6$.

6. $\dfrac{x+2}{x-1}-\dfrac{4-x}{2x}=\dfrac{7}{3}$.

7. $\dfrac{3a}{2a+5}-\dfrac{2}{a-1}=1$.

8. $\dfrac{2-3t}{4}-\dfrac{4-t}{t-2}=\dfrac{11}{4}$.

9. $\frac{3a-5}{2a-5}-\frac{2a+5}{3a+5}=1.$

10. $\frac{4v-9}{4v-3}-\frac{2v-3}{2v}=9.$

11. $\frac{2w+3}{5}-\frac{6-w}{w-4}=2.$

12. $\frac{3r-1}{7-r}-\frac{5-4r}{2r+1}=3.$

13. $\frac{1}{x-6}-\frac{4}{3(x-1)}=-\frac{2}{3x}.$

14. $\frac{y+2}{y-2}+\frac{y-2}{y+2}=\frac{6y+16}{3y}.$

15. $\frac{1}{1-t^2}+\frac{1}{1+t}-\frac{1}{1-t}=-\frac{7}{8}.$ $8(1-t)$

16. $\frac{3r-6}{5-r}=\frac{7}{2}-\frac{11-2r}{2(5-2r)}.$

17. $\frac{3-2x}{2+x}-\frac{2+3x}{2-x}=\frac{1}{3}+\frac{16x+x^2}{x^2-4}.$

18. $\frac{y+1}{y-1}-\frac{y+2}{y-2}=\frac{2y+13}{y-1}.$

19. $1+\frac{7x}{3x+1}+\frac{2x^2}{(3x+1)(7x+1)}=0.$

20. $\frac{1}{(3x+1)(1-5x)}-\frac{15x}{2(1-5x)(7x+1)}=\frac{x}{(3x+1)(7x+1)}.$

21. Solve the equation $2p^2x^2-3pqx-q^2=0.$

SOLUTION: 1. $a=2p^2$; $b=(-3pq)$; $c=(-q^2)$.

2. Formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$

$$x=\frac{-(-3pq)\pm\sqrt{(-3pq)^2-4(2p^2)(-q^2)}}{2(2p^2)}=\frac{+3pq\pm\sqrt{9p^2q^2+8p^2q^2}}{4p^2}$$

$$=\frac{+3pq\pm\sqrt{17p^2q^2}}{4p^2}=\frac{3pq\pm pq\sqrt{17}}{4p^2}=\frac{pq(3\pm\sqrt{17})}{4p^2}=\frac{q(3\pm\sqrt{17})}{4p}$$

Solve the following equations for x:

22. $x^2 - 6cx + 5c^2 = 0$.
23. $3x^2 = 2rx + 2r^2$.
24. $x^2 + c^2 - 2xp = 0$.
25. $x^2 - xp + (c^2 - d^2) = 0$.
26. $\frac{1}{2}gx^2 + ax = s$.
27. $2P - 2Px + x^2 = 1$.
28. $x^2 - (a-1)x = a$.
29. $x^2 + ax + bx + ab = 0$.
30. $ax^2 - (a - 2b)x = +2b$.
31. $mx^2 + (2mn - 3n)x - 6n^2 = 0$

EXERCISE 124

Review § 112 before solving this set of examples.

1. Twice the square of a certain number equals the excess of 3 over that number. Find the number.

2. If three times the square of a certain number be increased by the number itself, the sum is 10. Find the number.

3. Find two consecutive integers whose product is 306.

4. If the product of three consecutive integers be divided by each of them in turn, the sum of the three quotients is 74. Find the integers.

5. The sum of the squares of two consecutive integers is 685 Find the integers.

6. The sum of a certain number and its reciprocal is $\frac{13}{6}$. Find the number.

Hint: The reciprocal of a number is obtained by dividing 1 by the number. The reciprocal of x is $\frac{1}{x}$.

7. Find the dimensions of a rectangle whose area is 357 square feet if its length exceeds its width by 4 feet.

8. The numerator of a certain fraction exceeds its denominator by 3. The fraction exceeds its reciprocal by $\frac{39}{40}$. Find the fraction.

9. The main waiting room of the Union Railway Station in Washington, D.C., has an area of 28,600 square feet. The length exceeds the width by 90 feet. Find the dimensions.

Paul

10. Find the base and altitude of a triangle whose area is 63 square inches, if the base exceeds twice the altitude by 4 inches.

11. Find the dimensions of a rectangle whose area equals that of a square of side 24 feet, if the difference of the base and altitude of the rectangle is 14 feet.

12. Find the dimensions of a rectangle if its area equals that of a square of side 35 feet, if the difference of the base and altitude is 24 feet.

13. Find the dimensions of a rectangle whose area is 3750 square feet, if the sum of its base and altitude is 155 feet.

14. Find the dimensions of a rectangle whose area is 1701 square feet, if the sum of its base and altitude is 90 feet.

15. Find the dimensions of a right triangle if its hypotenuse is 20 feet and the base exceeds the altitude by 4 feet.

16. Find the dimensions of a right triangle if its hypotenuse is 26 feet and the sum of whose base and altitude is 34 feet.

17. Find the sides of an isosceles triangle if the perimeter is 35 inches and if the number of inches in the base is the quotient of 75 divided by the number of inches in one of the sides of the triangle.

18. A man travelled 105 miles. If he had gone 9 miles more an hour, he would have performed the journey in $1\frac{1}{2}$ hours less time. Find his rate in miles an hour. (See p 105)

19. If a man travels 120 miles by one train and returns on a train whose rate is 10 miles an hour more, he will require 7 hours for the trip. What is the rate of the first train?

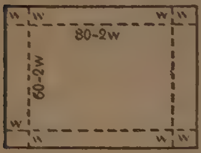
20. A crew can row 8 miles downstream and back again in $4\frac{4}{5}$ hours; if the rate of the stream is 4 miles an hour, find the rate of the crew in still water. (See § 144.)

21. A man travels 10 miles by train. He returns by a train which runs 10 miles an hour faster than the first, accomplishing the whole journey in 50 minutes. Find the rate of the first train.

22. A tank can be filled by two pipes running together in 2 hours The larger pipe by itself will fill it in 3 hours less time than the smaller pipe. How long will it take each pipe to fill the tank alone? (See § 142.)

23. Some boys are canoeing on a river, in part of which the current is 5 miles an hour, and in another part 3 miles an hour. If, when going downstream, they go 4 miles where the current is rapid and 8 miles where it is less rapid in a total time of $1\frac{5}{6}$ hours, what is their rate of rowing in still water?

24. I have a lawn which is 60 by 80 feet. How wide a strip must I cut around it when mowing the grass to have cut half of it?



HINT: Referring to the figure, it is clear that if w = the number of feet in the width of the border cut, then the dimensions of the uncut part of the lawn are $(60 - 2w)$ and $(80 - 2w)$.

Hence, $(60 - 2w)(80 - 2w) = \frac{1}{2} \cdot 60 \cdot 80$.

Complete the solution.

25. A farmer is plowing a field whose dimensions are 40 rods and 90 rods. How wide a border must he plow around the field in order to have completed $\frac{5}{6}$ of his plowing?

26. The numerator of a certain fraction is 8 less than the denominator If the denominator and numerator each be increased by 5, the resulting fraction is twice the fraction obtained by increasing the original denominator by 1. Find the fraction.

27. An automobile made a trip of 50 miles, 10 miles within the city limits and 40 miles outside the city limits Outside of the city, the rate was increased 15 miles an hour. If the trip took $2\frac{3}{5}$ hours, find the rate at which they travelled within and outside of the city limits.

28. The numerator of a certain fraction exceeds its denominator by 5. If the numerator be decreased by 3 and the denominator be increased by 4, the sum of the new fraction and the original fraction is 3. Find the original fraction.

29. A bicyclist rides a number of hours at a number of miles an hour which exceeds the number of hours by 3; an automobilist, starting 3 hours after him, overtakes him by going two and one half times as fast as he did. Find the rate of each.

30. The circumference of the fore wheel of a carriage is less by 4 feet than the circumference of the hind wheel. In travelling 1200 feet, the fore wheel makes 25 revolutions more than the hind wheel. Find the circumference of each wheel

to where feet /

IMAGINARY ROOTS IN A QUADRATIC EQUATION

199. Example. Solve the equation $x^2-2x+5=0$.

Solution: 1. Use the formula method of solving the equation.

$$a=1, \qquad b=-2, \qquad c=5.$$

2. $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{+2\pm\sqrt{4-4\cdot 1\cdot 5}}{2}$

3. $\quad =\dfrac{+2\pm\sqrt{4-20}}{2}=\dfrac{2\pm\sqrt{-16}}{2}.$

The question arises what does $\sqrt{-16}$ mean? Is -4 the square root of -16? No, for $(-4)^2=+16$. Is $+4$? No, for $(+4)^2=+16$. Thus, no number with which the student is acquainted will produce -16, when it is squared.

200. No number raised to an even power will produce a negative result; hence an even root of a negative number is impossible up to this point. To avoid this difficulty, a new kind of number is introduced.

An **Imaginary Number** is an indicated square root of a negative number; as $\sqrt{-16}$; $\sqrt{-3}$; $\sqrt{-a^3}$.

The numbers previously studied are called **Real Numbers.**

201. Every imaginary number can be expressed as the product of a real number and $\sqrt{-1}$.

$\sqrt{-1}$ is indicated by i, and is called the **Imaginary Unit.**

Thus, $\sqrt{-16}=\sqrt{16(-1)}=\pm\ 4\sqrt{-1}\ =\pm 4i.$

$\sqrt{-a^2}=\sqrt{a^2(-1)}=\pm\ a\sqrt{-1}\ =\pm ai.$

$\sqrt{-5}=\sqrt{5\ (-1)}=\pm\sqrt{5}\cdot\sqrt{-1}=\pm i\sqrt{5}.$

Historical Note. The symbol i for $\sqrt{-1}$ was introduced by Euler, one of the greatest mathematicians of the eighteenth century.

EXERCISE 125

Express the following in terms of i:

1. $\sqrt{-49}$. **2.** $\sqrt{-36}$. **3.** $\sqrt{-100}$. **4.** $\sqrt{-81a^2}$.

5. $\sqrt{-25}$. **6.** $\sqrt{-64}$. **7.** $\sqrt{-144}$. **8.** $\sqrt{-121a^2b^2}$.

9. $\sqrt{-\frac{1}{4}}$. **10.** $\sqrt{-\frac{4}{9}}$. **11** $\sqrt{-\frac{16}{25}}$. **12.** $\sqrt{-\frac{9}{49}}$.

13. $\sqrt{-3}$. **14.** $\sqrt{-32}$. **15.** $\sqrt{-18}$. **16.** $\sqrt{-72}$.

17. $\sqrt{-5}$. **18.** $\sqrt{-27}$. **19.** $\sqrt{-12}$. **20.** $\sqrt{-28}$.

21. Simplify $\sqrt{-\frac{27}{4}}$.

Solution: $\sqrt{-\frac{27}{4}}=\sqrt{\frac{9\cdot 3\cdot(-1)}{4}}=\pm\frac{3}{2}\sqrt{3}\cdot\sqrt{-1}=\pm\frac{3i\sqrt{3}}{2}.$

22. $\sqrt{-\frac{2}{9}}$.

23. $\sqrt{-\frac{3}{4}}$.

24. $\sqrt{-\frac{5}{36}}$.

25. $\sqrt{-\frac{8}{9}}$.

26. $\sqrt{-\frac{75}{4}}$.

27. $\sqrt{-\frac{20}{49}}$.

28. $\sqrt{-\frac{32}{25}}$

29. $\sqrt{-\frac{125}{64}}$.

30. $\sqrt{-\frac{63}{100}}$.

202. Addition and Subtraction of Imaginary Numbers.

EXERCISE 126

1. Add $\sqrt{-4}$ and $\sqrt{-36}$.

Solution: $\sqrt{-4}+\sqrt{-36}=2i+6i=8i$.

Note. While every imaginary number, like $\sqrt{-4}$, has two values, one positive and one negative, in problems such as the one in this exercise, only the *principal root*, the positive one, is used, as in the case of surds (§ 185).

2. $\sqrt{-16}+\sqrt{-25}$.

3. $\sqrt{-1}-\sqrt{-49}$.

4. $\sqrt{-81}-\sqrt{-64}$.

5. $\sqrt{-100}+\sqrt{-169}$.

6. $\sqrt{-1}+\sqrt{-49}-\sqrt{-64}$.

7. $\sqrt{-36}+\sqrt{-100}-\sqrt{-81}$.

8. $\sqrt{-a^2}-\sqrt{-4a^2}+\sqrt{-9a^2}$.

9. $\sqrt{-16x^2}-\sqrt{-25x^2}+\sqrt{-49x^2}$.

10. $\sqrt{-3}+\sqrt{-27}$.

11. $\sqrt{-18}-\sqrt{-8}$.

12. $\sqrt{-20}+\sqrt{-45}$.

13. $\sqrt{-28}--\sqrt{-7}+\sqrt{-63}$.

14. $\sqrt{-24}-\sqrt{-54}-\sqrt{-6}$.

15. $\sqrt{-44}+2\sqrt{-11}-\sqrt{-99}$.

16. Simplify $+\frac{5}{2}\pm\sqrt{-\frac{27}{4}}$.

Solution. 1.
$$\frac{5}{2}\pm\sqrt{-\frac{27}{4}}=\frac{5}{2}\pm\frac{\sqrt{-27}}{2}=\frac{5}{2}\pm\frac{3\sqrt{3}\cdot\sqrt{(-1)}}{2}$$
$$=\frac{5}{2}\pm\frac{3i\sqrt{3}}{2}$$
$$=\frac{5\pm3i\sqrt{3}}{2}$$

The numbers in Examples 1–15 are called **Pure Imaginaries.** The sum, or difference, of a pure imaginary and a real number, § 200, as in this exercise, is called a **Complex Number.**

Simplify the following:

17. $\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$.

18. $\frac{2}{3} \pm \sqrt{-\frac{4}{9}}$.

19. $\frac{3}{4} \pm \sqrt{-\frac{5}{16}}$.

20. $\frac{3}{5} \pm \sqrt{-\frac{28}{25}}$.

21. $\frac{7}{10} \pm \sqrt{-\frac{27}{100}}$.

22. $\frac{6}{13} \pm \sqrt{-\frac{75}{169}}$.

203. Meaning of Imaginary Roots of a Quadratic on the Graph.

EXAMPLE. Consider the equation $x^2 + x + 2 = 0$.

SOLUTION: 1. Solve the equation by the formula.

$$a = 1;\ b = 1;\ c = 2.$$

$$x = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm \sqrt{-7}}{2} = \frac{-1 \pm i\sqrt{7}}{2}.$$

$$x_1 = \frac{-1 + i\sqrt{7}}{2};\ x_2 = \frac{-1 - i\sqrt{7}}{2}.$$

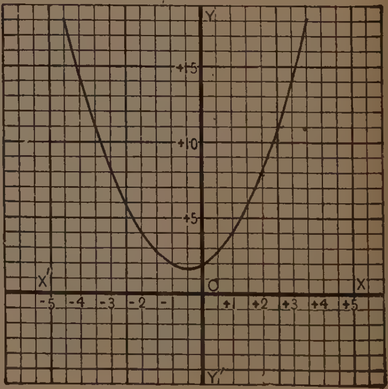
2. Solve the equation graphically. (Review rule § 193.)

Let $$y = x^2 + x + 2$$

When $x =$	0	+1	+2	+3	−1	−2	−3	−4
then $y =$	+2	+4	+8	+14	+2	+4	+8	+14

3. The graph has the same shape as the graphs obtained when solving other quadratic equations; *but the graph does not cross* the horizontal axis at all. Hence, y or $x^2 + x + 2$ is never zero for any real value of x.

This is characteristic of the graph of a quadratic which has *imaginary* roots.



EXERCISE 127

Solve the following equations. Express the roots in simplest form. Draw the graphs for the first three equations.

1. $x^2+2x+3=0.$
2. $x^2+3x+4=0.$
3. $2x^2-x+2=0.$
4. $3m^2-2m+5=0$
5. $5c^2-7c+3=0$
6. $8t^2=9t-5.$
7. $11r^2+6=15r.$
8. $\frac{x^2}{5}-\frac{3x}{10}+\frac{1}{2}=0.$
9. $\frac{2}{x+3}+\frac{3x}{5}=0.$
10. $\frac{2x+3}{x-1}=\frac{x+7}{x+4}.$
11. $\frac{3x}{2}+\frac{1}{x}+\frac{5}{3}=0.$
12. $2x^2-5ax+7a^2=0.$
13. $3x^2-4xw+2w^2=0.$
14. $5x^2-7xt+3t^2=0.$

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ANSWERS TO WELLS AND HART'S NEW HIGH SCHOOL ALGEBRA

EXERCISE 5. Page 11.

18. 73.09 lb. 19. $214\frac{2}{7}$ lb. 20. 150 lb.

EXERCISE 7. Page 14.

16. C, \$16; B, \$32; A, \$48. 17. \$839.
18. 8; 40; 72. 19. 12; 168; 180.
20. 2d, \$75; 1st, \$150; 3d, \$300. 21. 60 in.; 80 in.; 100 in.
22. 22 in.; 44 in. 23. 40 rd.; 120 rd.
24. AB, 20 in.; CD, 40 in.; AD, 60 in.; BC, 100 in.
25. Philadelphia, 90 mi.; Chicago, 900 mi.

EXERCISE 10. Pages 20–22.

1. *b*. Equal to it. *c*. A rectangle; b; a; ab. *d*. ab square units. *e*. The area of a parallelogram equals the product of the base and altitude. *g*. 240. *h*. 375. *i*. 12.5. *j*. 24. *k*. 50. *l*. 30.

2. *b*. A parallelogram; b; a; ab. *c*. $\frac{1}{2}$ of it. *d*. $\frac{1}{2}ab$. *e*. The area of a triangle equals one half of the product of the base and altitude. *g*. 85. *h*. 300. *i*. 13. *j*. 40.

3. *a*. $V = abc$. *b*. 120. *c*. 378. *d*. 8.

4. *a*. $V = \frac{1}{3}ab$. *b*. 90. *c*. 20. *d*. 90.

5. *a*. The circumference of a circle equals twice the product of π and the radius of the circle. *b*. 62.832 inches. *c*. 100 inches.

6. *a*. The area of a circle equals π times the square of the radius. *b*. 314.16 sq. in. *c*. 78.54 sq. ft.

7. 294.72. 8. 2915.1. 9. 113.1 cu. ft.
10. 314.16 sq. ft.

EXERCISE 16. Page 33.

14. -6. 15. 4. 16. -40.
17. -132. 18. -8. 19. 2.
20. -20. 21. -24. 22. -62.
23. -16. 24. -15. 25. -11.

EXERCISE 19. Pages 40–41.

1. $A - 2B$.
2. $-2t^2 + 6n^3$.
3. $-am + 4bn$.
4. $-b^2$.
5 $-4r^3 + r^2s + 9rs^2 - 10$.
6. $-2x^3 + x^2y - 5xy^2 + y^3$.
7. $-7m^3 - 2m^2n - 5mn^2 + 2n^3$.
8. $8a + 3b + 3c$.
9. $12k - 7 - m$.
10. $-2a + 5b - 2c + 9d$.
11. $35r + 4s - 4t$.
12. $10xy$.
13. $10n^2$.
14. $2a^3 + 6ab^2$.
15. $6x^2 + xy - 7y^2$.
16. $12a^3 + 24a^2 - 9a - 8$.
17. $2x^5 + 2x^3y^2 + x^2y^3 + 2xy^4$.
18. $19x^3 + 8x^2 - 23x - 7$.
19. $3A^3 + 11A^2B + 8AB^2 + 6B^3$.
20. $\frac{3}{4}a + \frac{1}{6}b - \frac{1}{10}c$.
21. $2\frac{1}{3}m - \frac{1}{5}n$.
22. $2\frac{1}{4}a - \frac{3}{8}b + \frac{1}{14}c$.
23. $.7A - 10.4B + 1.5C$.
24. $.75x^2 - .75xy + .8y^2$.
25. $3.5m + .7n + 2.4$.
26. $5a^2 - 8ab - 2b^2$.
27. $3x^3 - x - 4$.
28. $x^3 - 8x^2y - 2xy^2 - 3y^3$.
29. $3a^3 - 5a^2 + 4a - 2$
30. $a^3 + 21x^3$.

EXERCISE 22. Pages 45–47.

1. $4a^2 - 3a - 20$.
2. $8a - 15x + 5y$.
3. $-3m + 4n + 11p$.
4. $a + 3b$.
5. $3ab - 6bc + ac$.
6. $4x^2 + 7x - 6$.
7. $7r^3 + 2r^2s - 2rs$.
8. $-x^3 + 3x^2 - 6x + 5$.
9. $6a^2b - ab^2 + 3b^3$.
10. $x^4y - xy^4$.
11. $-4xy$.
12. $2x^2y + 2y^3$.
13. $-6b + 8c$.
14. $-10m^2 + 3mn - 5n^2$.
15. $2bc$.
16. $x^3 - x^2 + 6x - 7$.
17. $-3x + 3y - 3z$.
18. $4B - 14C$.
19. $7x^3 + 11x^2 + 2x - 5$.
20. $2y^2$.
21. $6a - 12b + 21c + 2d$.
22. $-9a^3 + 8a^2 - 4a + 3$.
23. $4a - 2a^2 - 2a^3$.
24. $10x^3 - 6x^2 + 9x - 12$.
25. $9a^3 + 3a^2b - 12ab^2 - 8b^3$.
26. $7a - b - 8c - 4d$.
27. $5 - 4x + 7x^2 - 20x^3 - 5x^4$.
28. $4a^5 + 10a^4 - 11a^3 - 16a^2 - 8a + 1$.
29. $x^5 - 5x^4y + 6x^3y^2 + 11x^2y^3 - 15xy^4 + y^5$.

30. $25n^3 - 21n^2 + 3n - 4$.

31. $2x^3 - 5x^2y + xy^2$.

32. $-26x^3 + 2x^2 + x - 5$.

33. $\frac{1}{4}m + n - \frac{1}{14}p$.

34. $1\frac{1}{6}a - \frac{5}{12}b - \frac{7}{10}c$.

35. $\frac{11}{15}v + \frac{1}{2}w + \frac{3}{14}x$.

36. $56b^2 - 6ab - 5a^2$.

37. $5a - 7b + 4c$.

38. $3a - 2b + c$.

39. $5a + 4b - 5$.

40. $4a^2 - 4ab$.

41. $2a^2 - 6ab + 10b^2$.

42. $-5x^3 + 10x^2 - 2x - 1$.

43. $x - 12y - z$.

44. $-5a + 9b - 3c - 3d$.

45. $2.5x - 2y + 2z$.

EXERCISE 24. Pages 51-52.

7. James, 15 yr.; John, 30 yr.

8. A, 20 yr.; B, 10 yr.; C, 30 yr

9. 35°; 140°.

10. 71 rd.

11. 32,000 sq. mi.

12. 20,400 ft.

13. Nile, 3500 mi.; Mississippi, 4200 mi.

14. Highest, 102 mi. per hr.; lowest, 4 mi. per hr.

15. Mobile, 62 in.; Yuma, 3.1 in.

EXERCISE 25. Page 54.

15. $37,500.

16. $3\frac{3}{4}$ yr.

17. 4%.

18. 1.04 P; 1.06 P; 1.07 P.

19. 1.08 P dollars; 1.15 P dollars.

20. $2800.

21. $6\frac{2}{3}$ yr.

22. $16\frac{2}{3}$ yr.

24. 6 yr.

25. 23.8 + %.

EXERCISE 26. Page 58.

21. $6 - 2c$.

22. $-3a$.

23. $-a + 3b$.

24. $2a$.

25. $3x - 2$.

26. $-b - 16$.

27. $3x$.

28. $7x^2 - x + 6$.

29. $-r + s$.

30. $-4q$.

31. $8m - 7n$.

32. $13r - 2s - t$.

33. $-2x - y$.

34. $4a - 1$.

35. $3b + 11$.

36. $12r$.

37. $5x + 12a - 4$.

38. $6ab + 17$.

39. $-4c + 2$.

40. $-6m + n$.

EXERCISE 29. Pages 62–64.

1. 17.
2. 20.
3. 18.
4. 2.
5. – 19.
6. – 8.
7. 1.
8. 9.
9. 1.
10. 0.
11. 13, 17.
12. A, 51 yr.; B, 64 yr.
13. 28, 95.
14. A, 64 yr.; B, 38 yr.
15. 58, 35.
16. 14 ft.; 21 ft.; 29 ft.
17. 840 ft.
18. $3800, gold leaf, $5000, appropriation of 1800.
19. 2, nearer; 5, farther.
20. 17, 18.
21. 35, 36, 37.
22. 53, 54, 55, 56.
23. 97, 99.
24. 76.
25. 42°, 126°, 116°, 76°.
26. 105°, angle; 75°, supplement.
27. 35°, angle; 55°, complement.
28. 75°.
29. Nitrogen, 20 lb.; phosphoric acid, 30 lb.; potash, 50 lb.
30. Rhode Island, 1248 sq. mi.; Texas, 265,896 sq. mi.

EXERCISE 34. Page 71.

15. $7x^2 - 35x$.
16. $-60a^3b + 42ab^3$.
17. $x^6y - 4x^4y^3 + x^2y^5$.
18. $-r^3s + r^2s^2 - rs^3$.
19. $-3a^3b + 6a^2b^2 - 3ab^3$.
20. $48x^8 - 40x^6 - 96x^5$.
21. $-12a^5b^3 + 8a^4b^4 + 16a^3b^5$.
22. $-40m^8 + 5m^6 + 15m^4$.
24. $50a + 7b$.
25. $-5ay$.
26. $x^2 - 2xy - y^2$.
27. $16m + 12$.
28. $-6cm - 36c$.
29. $r^3 - s^3$.
30. $x + 1$.
31. $-m$.
32. $18r - 5$.
33. $2 - 20s$.
34. $x^3 - 2x + 3$.
35. $3x^2 - xy + 2y^2$.
36. $t + 7$.
37. $14x - 10$.
38. $-3x + 2$.
39. $-7m$.
40. $-x^6 + 8x^4 - 5x^2$.
41. $\frac{1}{8}m^3n - \frac{1}{12}m^2n^2 + \frac{1}{15}mn^3$.

EXERCISE 35. Pages 73–74.

1. $x^2 + 8x + 15$.
2. $r^2 - 11r + 28$.
3. $2s^2 - 11s + 15$.
4. $3m^2 - 10m - 8$.
5. $4t^2 + 3t - 27$.
6. $6x^2 + 23x + 21$.
7. $10m^2 + 23m - 5$.
8. $12p^2 + 36p - 21$.
9. $30y^2 - 46y + 8$.
10. $28z^2 + 5z - 50$.
11. $2a^2 + 3ab + b^2$.
12. $6c^2 - cd - 2d^2$.
13. $15r^2 + 8rs - 12s^2$.
14. $15x^2 - 26xy + 8y^2$.
15. $24m^2 + 18mp - 15p^2$.
16. $42y^2 + 2yz - 72z^2$.
17. $66a^2 - 14ad - 20d^2$.
18. $96p^2 - 28pq - 49q^2$.
19. $2x^4 - 7x^2y^2 + 3y^4$.
20. $99w^4 - 50w^2v - 21v^2$
21. $m^3 + 2m^2 - 6m - 9$.
22. $10a^3 + 33a^2 - 52a + 9$.
23. $x^3 - 5x^2y + 9xy^2 - 9y^3$.
24. $x^3 + y^3$.
25. $x^3 - 64y^3$.
26. $m^4 + m^2n^2 + n^4$.
27. $a^5 - 6a^3 + 6a^2 - 7a + 6$.
28. $2a^4 - 7a^3 - 18a^2 - a + 3$.
29. $6x^4 + 13x^3 - 70x^2 + 71x - 20$.
30. $8n^4 - 50n^2 + 32$.
31. $63r^4 + 114r^3y + 49r^2y^2 - 16ry^3 - 20y^4$.
32. $12a^4 - 47a^3b - 8a^2b^2 + 107ab^3 + 56b^4$.
33. $a^2 - 2ab + b^2 - c^2$.
34. $r^2 - s^2 - 2st - t^2$.
35. $4n^4 - 5m^2n^2 + m^4$.
36. $a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5$.
37. $m^5 + 243$.
38. $8a^2 + 40ac - 18b^2 + 50c^2$.
39. $x^5 - 32$.
40. $a^3 - 3abc - b^3 - c^3$.
41. $\frac{1}{9}a^2 - \frac{1}{16}b^2$.
42. $\frac{1}{4}m^2 - \frac{1}{9}n^2$.
43. $\frac{1}{16}x^2 - \frac{1}{81}y^2$.
44. $2a^2 + \frac{2}{3}ab - \frac{1}{6}b^2$.
45. $x^2 - \frac{145}{72}xy + y^2$.
46. $9x^2 - 30x + 25$.
47. $8m^3 - 36m^2n + 54mn^2 - 27n^3$.
48. $64r^3 + 240r^2 + 300r + 125$.
49. $16x^4 - 96x^3 + 216x^2 - 216x + 81$.
50. $27a^3 - 108a^2b + 144ab^2 - 64b^3$.
51. $a^3 + a^2 - 14a - 24$.
52. $2m^3 - 5m^2 - 37m + 60$.
53. $x^6 - y^6$.
54. $m^4 - n^4$.
55. $x^6 - y^6$.

EXERCISE 36. Page 75.

1. $11\,a^2 - 111.$ 2. $12\,m.$
3. $2\,a^2 - 12\,x^2.$ 4. $29\,b^2 - 28\,ab.$
5. $h^2 + 3\,h + 18.$ 6. $2\,x^2 - 12\,x - 2.$
7. $12\,x^2 - 3\,x - 6.$ 8. $10\,a^2 - 76\,a - 36.$
9. $2\,x^2 + 79\,x + 22.$ 10. $2\,ab^2 + 2\,a^2b.$

EXERCISE 37. Pages 76–77.

1. 13. 2. 7. 3. 9.
4. 1. 5. 2. 6. 2.
7. 3. 8. 4. 9. 6.
10. 5. 11. 7. 12. 9.
13. − 13. 14. 2. 15. 10.
16. 32, 43. 17. 42, 58.
18. Paris, 4020 mi.; London, 3740 mi.
19. 40, 47. 20. 25, 35. 21. 24, 56.
22. Length, 470 ft.; width, 340 ft.
23. 15, 16. 24. − 21, − 20.
25. Chicago, 2,185,283; Philadelphia, 1,549,008; New York, 4,766,883

EXERCISE 39. Pages 79–81.

1. A, 45 yr.; B, 5 yr. 2. Father, 27 yr.; son, 3 yr.
3. Father, 40 yr.; son, 15 yr. 4. A, 45 yr.; B, 9 yr,
5. A, 42 yr.; B, 84 yr. 6. 8 yr. ✓
7. 17 yr. ✓ 9. 15 lb. at 70 ¢; 35 lb. at 40 ¢.
10. $31\frac{1}{4}$ lb. at 36 ¢; $68\frac{3}{4}$ lb. at 20 ¢.
11. $33\frac{1}{3}$ lb. clover seed; $166\frac{2}{3}$ lb. blue grass seed.
12. 6 quarters; 13 dimes.
13. 5 half-dollars; 15 dimes.
14. 7 quarters; 9 nickels.
15. 8 \$2 bills; 13 fifty-cent pieces; 24 dimes.

EXERCISE 40. Page 82.

1. 5.
2. 4.
3. 2.
4. 12.
5. 2.
6. 10.
7. -21.
8. -20.
9. 7.
10. 3.
11. 18.
12. 10.
13. 4.
14. $\frac{1}{2}$.
15. $\frac{3}{5}$.
16. 6.
17. $\frac{3}{4}$.
18. $\frac{3}{2}$.
19. 13.
20. $\frac{5}{7}$.

EXERCISE 41. Pages 83–84.

1. 40.
2. 15, 20, 30.
3. 30.
4. 42.
5. 56.
6. 750.
7. 20.
8. 105.
9. 90°, 60°, 30°.
10. 90°, 45°, 45°.
11. 12, 13, 14.
12. 12 dimes, 3 dollars, 15 cents.
13. Length, 450 ft.; width, 250 ft.
14. Michigan, 1800 ft.; Huron, 1000 ft.; Superior, 900 ft.
15. Length, 760 ft.; width, 130 ft.
16. 1820, 9.6 million; 1910, 92.2 million.
17. Phosphoric acid, 12 lb.; potash, 60 lb.; nitrogen, 27 lb.
18. Length, $751\frac{1}{3}$ ft.; width, 350 ft.
19. $20.50.
20. Water, 3740 mi.; rail, 3250 mi.

EXERCISE 46. Pages 94–95.

15. $n^2 - 10$.
16. $x - 12a$.
17. $a - 6b$.
18. $x + 5z$.
19. $x^2 + 8y$.
20. $xy - 12$.
21. $5x - 7$.
22. $3a - 7$.
23. $2a - 3$.
24. $5x - 8$.
25. $8x - 3y$.
26. $5m + 4n$.
27. $x^2 + x - 12$.
28. $3a^2 + 2a - 4$.
29. $2x^2 - 3xy + 4y^2$.
30. $3a^2 - 2ab - b^2$.

31. $x^2 + 2xy + y^2$.
32. $n + 2$; remainder, 2.
33. $4x - 3$.
34. $a^2 + a - 1$; remainder, 20.
35. $m^2 - 3m - 4$.
36. $x^2 + xy + y^2$.
37. $x^2 - 2x + 4$.
38. $x^3 + 2x^2 + 4x + 8$.
39. $x^3 - x^2y + xy^2 - y^3$.
40. $x^3 + x^2y + xy^2 + y^3$.
41. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
42. $x^4 + x^3y + x^2y^2 + xy^3 + y^4$; rem. $2y^5$.
43. $x^4 - 2x^3 + 4x^2 - 8x + 16$.
44. $1 - 2a + 4a^2 - 8a^3$.
45. $n - 2$.
46. $2x^2 + 9x - 5$.
47. $4m^2 - 2mn^2 + n^4$.
48. $7x^2 + 8x + 4$; rem. 40.
49. $x^3 + 2x^2 - 6x + 5$.
50. $4a^2 + ab - 5b^2$.
51. $x - \frac{1}{2}$.
52. $x - 3$.
53. $3x + \frac{1}{3}$.
54. $\frac{1}{2}a + 2$.

EXERCISE 47. Pages 99-100.

1. 7.
2. -5.
3. 1.
4. 3.
5. -3.
6. 5.
7. 2.
8. 4.
9. $\frac{4}{5}$.
10. $\frac{2}{3}$.
11. -5.
12. -10.
13. $-\frac{2}{7}$.
14. $\frac{1}{2}$.
15. 2.
16. $-\frac{7}{5}$.
17. -1.
18. 6.
19. -12.
20. 7.

EXERCISE 48. Pages 100-103.

1. 31, 13.
2. -5.
3. 9, 10.
4. 24, 14.
5. Father, 35 yr.; son, 7 yr.
6. 15, 17, 19.
7. A, $7.50; B, $5.25; C, $9.25.
8. 35, 14.
9. Width, 430 ft.; length, 780 ft.
10. 10 in.; 20 in.; 15 in.
11. 27, 107.
12. 278 ft. below sea level.
13. B, $36; A, $12.
14. 8 quarters, 5 nickels, 16 half-dollars, 13 dimes.
15. Pink variety, 300; yellow variety, 200; scarlet variety, 800.
16. 55°, 55°, 70°.

17. 285 in.; 285 in.

18. 55° below zero.

19. Virginia, 42,627 sq. mi.; Nebraska, 77,520 sq. mi.; California, 158,297 sq. mi.

20. Girls, 516,536; boys, 398,525.

22. $700, at 5%; $2000, at 6%.

23. $5000.

24. $9000.

25. U. S. Steel Stock, $3000; Chicago Edison Bonds, $15,000.

EXERCISE 49. Pages 105–106.

11. $7\frac{1}{2}$ hr.

12. 9 hr.

13. $12\frac{1}{2}$ hr.

14. 6 hr.

15. 21 mi. per hr.

16. 15 mi. per hr.

17. A, 29 mi. per hr.; B, 21 mi. per hr.

18. 6 mi.

19. $45\frac{5}{17}$ mi.

20. 5 hr.

21. 24 mi. per hr.

22. $33\frac{1}{3}$ mi. an hr.

23. 1 hr. 52 min.

EXERCISE 50. Pages 108–109.

1. a, 27°; b, 45°; c, 90°; d, 180°.

2. b, 86°; c, 131°; d, 14°.

5. $459\frac{2}{5}$° below zero.

6. Tin, $449\frac{3}{5}$°; iron, 2192°; mercury, $-38\frac{1}{5}$°; paraffine, 131°.

7. Fahrenheit, $\frac{7}{5}$°; Faraday, $-151\frac{3}{5}$°; Dewar, $-439\frac{3}{5}$°; Onnes, $-452\frac{1}{5}$°.

8. a, 10°; b, $-11\frac{1}{9}$°; c, $-22\frac{2}{9}$°.

9. Alcohol, 78°; turpentine, 160°.

10. $-295\frac{3}{5}$°.

EXERCISE 63. Pages 134–135.

1. $2x^2 + 8x + 6$.

2. $2x^2 + 7x + 6$.

3 $2x^2 + 7x + 3$.

4. $6x^2 + 11x + 3$.

5 $6x^2 + 13x + 6$.

6. $6a^2 - 7a + 2$.

7. $3a^2 - 10a + 8$.

8. $3a^2 - 13a + 12$.

9. $6a^2 - 19a + 15$.
10. $8a^2 - 30a + 25$.
11. $6r^2 + 7r - 5$.
12. $8r^2 + 2r - 21$.
13. $20r^2 + 6r - 8$.
14. $21r^2 + 4r - 12$.
15. $32r^2 + 12r - 27$.
16. $18s^2 - 3s - 10$.
17. $40s^2 - 18s - 7$.
18. $55s^2 - 16s - 48$.
19. $36s^2 - 3s - 60$.
20. $45s^2 - 30s - 40$.
21. $18a^2b^2 - 24ab - 10$.
22. $42m^2n^2 - 45mn + 12$.
23. $72r^4 + 5r^2 - 12$.
24. $65x^6 - 4x^3 - 21$.
25. $24p^4 - 10p^2 - 21$.
26. $55m^4 + 46m^2 - 24$.
27. $54c^6 - 3c^3 - 2$.
28. $150n^6 + 5n^3 - 12$.
29. $28x^4 + 4x^2 - 5$.
30. $48t^6 + 5t^3 - 18$.
31. $70r^2s^2 - 3rs - 27$.
32. $8a^2 - 2ab - 15b^2$.
33. $42m^2 + 11ms - 20s^2$.
34. $72t^2 - 35tx + 3x^2$.
35. $x^2 - 25y^2$.
36. $4x^2 - 9y^2$.
37. $4m^2 - 20mn + 25n^2$.
38. $9t^2 + 24nt + 16n^2$.
39. $27a^2b^2 + 33abc - 20c^2$.
40. $30x^2y^2 + xyz - 42z^2$.
41. $56r^4 + r^2s - 72s^2$.
42. $110x^6 - x^3y^3 - 132y^6$.
43. $54m^2n^4 - 3mn^2 - 12$.
44. $6 + 31x + 18x^2$.
45. $30 + 3t^2 - 63t^4$.
46. $80 - 2z^3 - 12z^6$.
47. $35 + 12xy - 36x^2y^2$.
48. $36 - 53xy^2 + 11x^2y^4$.
49. $90a^4b^2 - 74a^2bc + 12c^2$.
50. $60x^4 - 13x^2y - 28y^2$.
51. $63t^4 + 10t^2x^2 - 25x^4$.
52. $32m^4 + 4m^2n - 45n^2$.
53. $35x^4 + x^2y^2 - 6y^4$.
54. $44c^4 - 75c^2d + 25d^2$.
55. $135p^4 - 12p^2q - 4q^2$.
56. $160a^2 + 4ab - 21b^2$.
57. $90m^2 + 63mn - 10n^2$.
58. $24m^2 + 4m + \frac{1}{6}$.
59. $72x^2 - 10xy + \frac{1}{3}y^2$.
60. $120a^2 - \frac{4}{3}ab - \frac{4}{9}b^2$.

EXERCISE 65. Page 138.

9. 8.
10. $3\frac{1}{3}$.

EXERCISE 68. Pages 142–143.

1. $12x^2 - xy - 6y^2$.
2. $\frac{9}{121}x^2 - 36y^2$.
3. $a^6 - 1$.
4. $4a^2 - 4ab^2 + b^4$.
5. $21a^4 - 53a^2b^2 - 8b^4$.
6. $a^4 - b^4$.

7. $9x^2 + 7xy - 2y^2$.
8. $3a^2x - 3b^2x$.
9. $25a^2 - 50ab^2 + 24b^4$.
10. $3m^5 - 12m^3n^2 + 6mn^4$.
11. $7n^7 - 7n$.
12. $3a^3 - 3a$.
13. $a^4 - 3a^2y - 108y^2$.
14. $x^6 - y^6$.
15. $x^2 - \frac{3}{2}xy + \frac{9}{16}y^2$.
16. $m^2 - \frac{3}{4}m + \frac{1}{8}$.
17. $6x^3 - 22x^2 - 8x$.
18. $B^4 - 10AB^2C + 24A^2C^2$.
19. $a^7 - 9a$.
20. $10a^2 + 5ab - 15b^2$.
21. $30y + 3y^2 - 9y^3$.
22. $21m^2 - 119m - 42$.
23. $x^6 - y^4$.
24. $x^2y^2 + 12xy - 64$.
25. $2x^2 - 16xy + 32y^2$.
26. $x^4 - 4x^2 + 3$.
27. $x^3 + \frac{1}{8}$.
28. $\frac{4}{25}x^4 - \frac{9}{16}$.
29. $4n^2 - 2n + \frac{1}{4}$.
30. $x^2 - \frac{1}{8}x - \frac{15}{32}$.

EXERCISE 69. Page 145.

13. 11, 9.
14. 3, 4, 5, 6.
15. 7, 8.
16. 125, 126.
17. 6 mi. per hr.; 6 hr.

EXERCISE 70. Pages 145–147.

1. *a.* 600 sq. in.; *b.* 300 sq. in.; *c.* 600 sq. in.

2. *a.* $2x(x-5)$ sq. in.; *b.* $x(x-5)$ sq. in.; *c.* $2x(x-5)$ sq. in.

3. *a.* $y(y+4)$ sq. in.; *b.* $\frac{y(y+4)}{2}$ sq. in.; *c.* $y(y+4)$ sq. in.

4. *a.* s^2 sq. in.; *b.* base, $(s+4)$ in.; altitude, $(s-3)$ in.; *c.* $(s+4)(s-3)$ sq. in.; *d.* $(s+4)(s-3) = s^2 + 50$.

5. *a.* Altitude, a in.; base, $(2a+5)$ in.; area, $a(2a+5)$ sq. in. *b.* altitude, $(a+3)$ in.; base, $(2a+9)$ in.; area, $\frac{(a+3)(2a+9)}{2}$ sq. in.; *c.* $a(2a+5) = \frac{(a+3)(2a+9)}{2} + 25$.

6. Altitude, 6 in.; base, 14 in.

7. Altitude, 79 ft.; base, 96 ft.

8. Width, 40 ft.; length, 50 ft.

9. Upper square, 34 ft.; lower square, 55 ft.

10. 144 trees.

EXERCISE 71. Page 149.

1. $4, 8.$
2. $11, -5.$
3. $7, -9.$
4. $6, 12.$
5. $10, 11.$
6. $1, -2.$
7. $1, 6.$
8. $7, -3.$
9. $+\frac{2}{3}, -\frac{2}{3}.$
10. $+\frac{5}{6}, -\frac{5}{6}.$
11. $0, 5.$
12. $0, \frac{2}{3}.$
13. $1, -\frac{3}{8}.$
14. $\frac{3}{2}, -\frac{7}{2}.$
15. $\frac{7}{6}, -\frac{5}{4}.$
16. $\frac{1}{2}, \frac{3}{2}.$
17. $-\frac{1}{2}, -\frac{2}{3}.$
18. $-5, \frac{2}{3}.$
19. $3, \frac{1}{3}.$
20. $-2, -\frac{1}{2}.$

EXERCISE 72. Pages 150–151.

1. $-7a, 5a.$
2. $+\frac{3}{4}b, -\frac{3}{4}b.$
3. $-13m, -10m.$
4. $+\frac{3}{5}c, -\frac{3}{5}c.$
5. $-3a, -\frac{1}{2}a.$
6. $b, \frac{3}{7}b.$
7. $\frac{c}{2}, -\frac{5c}{3}.$
8. $-t, \frac{14t}{13}.$
9. $-v, \frac{13v}{9}.$
10. $\frac{k}{7}, -6k.$
11. $-a, 3a.$
12. $-10p, 2p.$
13. $3c, +12c.$
14. $4n, 12n.$
15. $8b, -\frac{b}{2}.$
16. $-\frac{m}{10}, \frac{5m}{3}.$
17. $c, -\frac{3a}{8}.$
18. $5t, -\frac{t}{3}.$
19. $-2r, \frac{11r}{3}.$
20. $-\frac{5s}{6}, \frac{3s}{2}.$

EXERCISE 73. Pages 152–153.

1. -3, or $\frac{5}{4}$.
2. $3, 4, 5; -1, 0, 1.$
3. $11, 7.$
4. $5, 2.$
5. Altitude, 3 ft.; base, 10 ft.
6. Length, $2\frac{1}{2}$ mi.; width, $\frac{1}{2}$ mi.
7. \$75, or \$25.
8. Length, 30 rd.; width, 10 rd.
9. Chicago lot: length, 125 ft.; width, 25 ft. Indianapolis lot: length, 130 ft.; width, 35 ft.
10. 5 ft.

EXERCISE 75. Page 156.

21. $m^2 + 2m + 4$.
22. $3a^2b$.
23. $a + 4$.
24. $x - 5y$.
25. $3a + 2b$.

EXERCISE 76. Pages 158–159.

5. $3 \cdot 5 \cdot 7 \cdot 11$.
6. $3 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 5$.
20. $6(r + s)(r - s)(r - t)$.
21. $(a - 4)(a - 3)(a - 5)$.
22. $(1 - x)^3(1 + x)$.
23. $3(2 - 3x)^2(2 + 3x)^2$.
24. $(2 - x)(3 - x)(4 - x)$.
26. $(r - 4)(r + 4)(r + 7)$.
27. $(a + x)^2(a^2 - ax + x^2)$.
28. $yz(2x - 1)(4x^2 + 2x + 1)(11x + 1)$.
29. $(b - 7)(b - 5)(b + 9)(b - 12)$.
30. $(2x - 5)^2(2x + 5)$.
31. $12(m - 6)^2(m + 8)(m + 4)$.
32. $(3n - 8)(3n - 1)(n + 2)$.
33. $(1+3x)(1-3x+9x^2)(1-8x)$.
34. $(x + 7)(x - 6)(x + 5)$.
35. $a(a^2 - 1)(a - 10)$.

EXERCISE 78. Page 164.

1. $\frac{2y + x}{3y - 2x}$.
2. $-\frac{3}{2x + y}$.
3. $-\frac{a + 4}{a + 6}$.
4. $-\frac{3(m + n)}{5m + n}$.
5. $\frac{x - 2}{5 - x}$.
6. $\frac{t^2 + 3t + 9}{t + 3}$.
7. $\frac{x + 4}{x^2 + 4x + 16}$.
8. $-\frac{r + s}{2(3r + s)}$.
9. $\frac{3c(d - c)}{5(d + c)}$.
10. $-\frac{2m(3x + 2y)}{3n(9x^2 + 6xy + 4y^2)}$.

EXERCISE 79. Page 166.

9. $3y + 1 + \frac{6}{3y - 1}$.
10. $4p - 6 + \frac{21}{2p + 3}$.
11. $x^2 + xy + y^2 + \frac{3y^3}{x - y}$.
12. $a^2 + 2ab + 4b^2 + \frac{16b^3}{a - 2b}$.
13. $x^3 + x^2y + xy^2 + y^3 + \frac{2y^4}{x - y}$.
14. $x^2 + 3xy + 6y^2 + \frac{18y^3}{x - 2y}$.
15. $3a + 2 + \frac{7}{4a - 1}$.
16. $3a + 14 + \frac{37a + 35}{a^2 - 2a - 3}$.

EXERCISE 81. Pages 172–175.

23. $\dfrac{2\,b^2 - ab + 3\,a^2}{a^2b^2}$.

24. $\dfrac{st + 2\,rt - 3\,rs}{rst}$.

25. $\dfrac{axz + byz - cxy}{xyz}$.

26. $\dfrac{5\,an^2 - 10\,bmn - 2\,cm^2}{10\,m^2n^2}$.

27. $\dfrac{14\,a - 5}{6\,a}$.

28. $\dfrac{3\,y^2 - 6\,x^2}{2\,x^2y^2}$.

29. $-\dfrac{3\,x + 4\,z}{6\,xz}$.

30. $\dfrac{ac + bc + ab}{abc}$.

33. $\dfrac{m^2 + 1}{m^2 - 1}$.

34. $\dfrac{2\,r^2}{r^2 - 9}$.

35. $\dfrac{13 - 4\,m}{(5\,m - 2)(2\,m + 3)}$.

36. $\dfrac{38\,b}{(3\,a - 4)(5\,a + 6)}$.

37. $\dfrac{2\,x^2 - 9\,y^2}{(3\,y - x)(2\,x - 3\,y)}$.

38. $\dfrac{3\,x^2 - xy + 2\,y^2}{6(x^2 - y^2)}$.

39. $\dfrac{29}{(3\,a - 7)(6\,a + 15)}$.

40. $\dfrac{29\,p - 12}{6(2\,p - 3)(5\,p - 4)}$.

41. $\dfrac{8\,x}{x^2 - 4}$.

42. $\dfrac{4\,p}{4\,p^2 - 1}$.

43. $\dfrac{2\,ab}{8\,a^3 + b^3}$.

44. $\dfrac{6\,x(x + 3)}{x^3 - 27}$.

45. $\dfrac{8}{x + 2}$.

46. 0.

47. $\dfrac{a^3 - 2\,x^3}{(a + x)(a^3 - x^3)}$.

48. 0.

49. $\dfrac{20 - 3\,x}{(x - 2)(x - 3)(x + 5)}$.

50. $\dfrac{a^2 + 2\,a - 2}{(a - 1)(a + 2)(a - 3)}$.

51. $\dfrac{2\,a^2 + 18\,a + 3}{(2\,a - 1)(a + 2)(3\,a + 1)}$.

52. $\dfrac{r^2 + 6\,ar + 3\,a^2}{(r - 3\,a)^2(r + 7\,a)}$.

53. $\dfrac{-11}{(x + 1)(x - 2)(x + 3)}$.

54. $\dfrac{a - 3}{a^2 - a + 1}$.

56. $\dfrac{2\,a + 15}{9 - a^2}$.

57. $\dfrac{x + y}{xy(y - x)}$.

58. $\dfrac{10\,x + 1}{12(x - 2)}$.

59. $\frac{4a}{m(16-m^2)}$.
60. $\frac{12}{1-a^3}$.
61. $\frac{r^2}{r^2-4}$.
62. $\frac{a-b}{a+b}$.
63. $\frac{x^2-x}{(x+5)(x-2)(x+3)}$.
64. $\frac{208}{15}$.
65. $\frac{3a^2-a+2}{3a}$.
66. $\frac{2x}{x+y}$.
67. $\frac{2b}{5a+b}$.
68. $\frac{6a^2-6}{2a-3}$.
69. 0.
70. $\frac{32}{4-3m}$.
71. $\frac{r^3-s^3}{r+s}$.

EXERCISE 82 Pages 176–177

3. $\frac{ay}{x^2}$.
4. $\frac{5b}{4a}$.
5. $\frac{1}{2}$.
6. 1.
7. $2abc$.
8. $\frac{9xy}{8mn^2}$.
9. $\frac{2(x-a)}{3x}$.
10. $\frac{2m-1}{m-4}$.
11. $\frac{a-5}{3a^2}$.
12. $\frac{3p-3r}{4(p-9)}$.
13. $\frac{2(a-3)}{(a-6)^2}$.
14. $\frac{x(x-1)}{(x+1)(x+2)}$.
15. $\frac{y^2(x-2y)}{x^2(x-y)}$.
16. $\frac{1}{3a}$.
17. $\frac{3r-2s}{3r+2s}$.
18. $\frac{m-3}{2(m-2)}$.
19. $\frac{5(x+y)}{2}$.
20. $\frac{2x-3}{3x-2}$.
21. $\frac{m+1}{2}$.
22. a.
23. $-\frac{8}{3}$.
24. $\frac{-2x}{a+x}$.
25. -1.

EXERCISE 83. Page 179.

7. $\frac{3(x+3)}{2(x-2)}$.
8. $\frac{r(2r+5v)}{v(4r-3v)}$.
9. $\frac{2c^2}{c-1}$.
10. $\frac{(c+d)^2}{c-d}$.
11. $\frac{4(r+s)}{r-s}$.
12. $\frac{t(t+2s)}{s}$.
13. $\frac{a(a+7)}{m^2(a-3)^2}$.
14. $\frac{w(w-2)}{w+5}$.
15. $\frac{(a+2b)(a-5b)}{(a+3b)(a+4b)}$.
16. $\frac{1}{m}$.
17. 1.
18. $\frac{a}{a-b}$.
19. $\frac{20r(r+s)(2r-3s)}{3s(2r+3s)(r+2s)}$.
20. $m-n$.

EXERCISE 84. Page 181.

1. 12.
2. 9.
3. $\frac{1}{x+y}$.
4. a.
5. $\frac{2b-3a}{2b+3a}$.
6. $\frac{ad-bc}{ad+bc}$.
7. $\frac{3b}{4b^2+2ab+a^2}$.
8. $\frac{3(2a-5b)}{5(4a+3b)}$.
9. $\frac{b+a}{b}$.
10. $2ab(a-2b)$.
11. $\frac{2x+y}{x+2y}$.
12. $\frac{x-1}{4x-3}$.
13. $\frac{a-b}{a+b}$.
14. $\frac{a+6}{a+1}$.
15. $\frac{2r^2}{r-s}$.
16. $\frac{3x+4}{x+1}$.
17. $\frac{3}{4}$.
18. $\frac{1}{vx}$.
19. $\frac{a}{a-1}$.
20. 3.

EXERCISE 85. Pages 183–184.

1. $-\frac{1}{17}$.
2. $\frac{3}{5}$.
3. $\frac{15}{4}$.
4. $\frac{38}{27}$.
5. $\frac{13}{5}$.
6. $\frac{x}{2a}$.
7. $\frac{3}{x-y}$.
8. $\frac{3x}{x+2}$.
9. $\frac{2}{2n-1}$.

10. $\frac{2}{2a-1}$. **11.** $\frac{p^2+p+1}{p}$. **12.** $\frac{1+x^2}{x}$.

13. $\frac{ad-bc}{ad+bc}$. **14.** $\frac{2x-3y}{6}$. **15.** $\frac{x+5}{x+2}$.

16. $\frac{x^2-x-6}{x}$. **17.** $\frac{a^2-4b^2}{a^2-b^2}$. **18.** $\frac{x-2y}{y}$.

19. $\frac{m(m^2+n^2)}{n(m^2+2mn-n^2)}$. **20.** $a-1$. **21.** $\frac{a-b}{a+b}$.

22. $\frac{x+3y}{3x-y}$. **23.** $\frac{3x-2}{x+1}$. **24.** x.

25. $\frac{2a^2+3b^2}{5ab}$.

EXERCISE 86. Pages 186-189.

1. 3. **2.** 2. **3.** 5. **4.** -2. **5.** 7. **6.** 6.
7. 15. **8.** 1. **9.** -5. **10.** 10. **11.** -3. **12.** -4
13. 3. **14.** -1. **15.** $\frac{1}{2}$. ~~**17.** 3.~~ **18.** 4. **19.** $\frac{1}{2}$.
20. 2. **21.** -4. **22.** 1. **23.** -1. **24.** $-\frac{1}{2}$. **25.** -5.
26. 3. **28.** 6. **29.** 23. **30.** 6. **31.** $\frac{4}{5}$. **32.** $\frac{1}{3}$.
33. -1. **34.** $-\frac{4}{5}$. **35.** $-\frac{1}{3}$. **36.** 1. **37.** $2\frac{1}{8}$. **38.** -1.
39. 2. **40.** 2. **41.** $-\frac{7}{10}$. **42.** -5. **43.** 3. **44.** $\frac{4}{3}$.
45 -2. **47.** -1. **48.** $\frac{4}{15}$. **49.** $\frac{3}{5}$. **50.** $\frac{1}{5}$.

EXERCISE 87. Pages 189-191.

1. 32, 24. **2.** Base, 13 ft.; altitude, 5 ft.; area, 65 sq. ft.
3 $25. **4.** 27, 18. **5.** $\frac{18}{45}$. **6.** A, 24 yrs.; B, 64 yrs.
7. Oklahoma, 1907; Washington, 1889. **8.** 79. **9.** 48.
10. $\frac{5}{11}$. **11.** 69°. **12.** 18 at 50¢ per dozen; 36 at 75¢ per dozen.
13 Engine, $18,700; sleeper, $19,000; parlor car, $15,500.
14. $2300 at 5%; $700 at 6%. **15.** Wyoming, 1890; Maine, 1820.
16. $\frac{42}{115}$. **17.** $2150 at 5%; $1800 at 4%. **18.** $\frac{3}{8}$. **19.** 30°. **20.** 1.

EXERCISE 88. Page 192.

5. $3\frac{3}{4}$ da. **6.** $3\frac{3}{5}$ da. **7.** $1\frac{1}{5}$ hr.
8. 9 hr. 36 min. **9.** 6 hr. **10.** $22\frac{1}{2}$ da.

EXERCISE 89. Pages 193–194.

1. 120 lb. 2. 8 ft. from the fulcrum.

3. 5 ft. from A. 4. $2\frac{2}{5}$ ft.

5. 7 ft. from the fulcrum.

6. On the left side, 7 ft. from the fulcrum.

EXERCISE 90. Pages 196–197.

4. *a.* $(r+5)$ mi. per hour;

b. slow train, $\frac{100}{r}$ hr.; fast train, $\frac{100}{r+5}$ hr.

5. *a.* $\frac{50}{r}$ hr., $\frac{250}{3r}$ hr.; *b.* $\frac{250}{3r}=\frac{50}{r}+1$.

6. *a.* Slow train, $\frac{100}{t}$ mi. per hour; fast train, $\frac{150}{t}$ mi. per hour.

b. $\frac{150}{t}=\frac{100}{t}+20$.

7. Slow train, 15 mi. per hour; fast train, 45 mi. per hour.

8. 12 mi. per hour.

9. Freight train, 15 mi. per hour.; passenger train, 35 mi. per hour.

10. 40 mi. per hour.

11. *a.* Downstream, 8 mi. per hour *b.* Upstream, 2 mi. per hour.

12. *a.* 3 hr., *b.* 12 hr.; *c.* 15 hr.

13. *a.* $\frac{d}{8}$ hr.; *b.* $\frac{d}{2}$ hr.; *c.* $\frac{d}{8}+\frac{d}{2}=5.$; *d.* 8 mi.

14. 12 mi. 15. $21\frac{3}{5}$ mi.

EXERCISE 91. Pages 198–200.

1. 20. 2. 21 yr.

3. Width, 36 ft.; length, 60 ft. 4. 30°, 15°, 60°, 75°, 180°.

5. 33°, 67°, 80°. 6. A, 32 mi.; B, 25 mi.

7. Shaft, 500 ft.; pyramid, 55 ft.

8. Slow train, 27 mi. per hour; fast train, 45 mi. per hour.

9. 10, 12, 14. 10. 10 da.

11. $4\frac{1}{2}$ mi. per hour. 12. A, 31 yr.; B, 12 yr.

13. $10,600.

14. Slow train, 30 mi. per hour; fast train, 45 mi. per hour.
15. 6 hr. 16. \$630 at $4\frac{1}{2}$%; \$810 at $3\frac{1}{2}$%.
17. 3 mi. per hour.
18. 3 5. 19. 2.65. 20. .90013.

EXERCISE 92. Pages 201-202.

1. $7a$. 2. $2b$. 3. $\frac{2a}{3b}$. 4. $\frac{3s}{2r}$.
5. c. 6. $a - b$. 7. $r + s$.
8. $c^2 - cd + d^2$. 9. $\frac{m-1}{m}$. 10. $m^2 - mn + n^2$.
11. $2b + a$. 12. $a - 1$. 13. $2m$. 14. $-a$.
15. $\frac{10m}{3n}$. 16. $12(a - b)$. 17. $m + n$. 18. $\frac{2}{n}$.
19. $a + b$. 20. $\frac{1}{a-b}$. 21. $-2a$. 22. $-\frac{2a}{3b}$.
23. $\frac{2a^2}{2a-b}$. 24. $\frac{b^2}{a}$. 25. $\frac{a}{2}$.

EXERCISE 93 Pages 204-205.

1. a. $\frac{A}{b}$; b. $\frac{A}{a}$. 2. a. $\frac{2A}{b}$; b. $\frac{2A}{a}$.
3. $\frac{C}{2\pi}$. 4. a. $\frac{V}{lh}$; b. $\frac{V}{lw}$.
5. a. $\frac{3V}{h}$; b. $\frac{3V}{b}$. 6. $C = \frac{5}{9}(F - 32)$.
7. a. 330; b. 20; c. $\frac{2A}{b+c}$; d. $\frac{2A-ac}{a}$.
8. a. $P = \frac{100A}{100+rt}$; b. $t = \frac{100(A-P)}{Pr}$.
9. a. $\frac{aT-1}{a}$; b. $\frac{1}{T-t}$.
10. a. $m(g-f)$; b. $\frac{mg-T}{m}$; c. $\frac{T}{g-f}$.
11. a. $\frac{Mu}{v-u}$; b. $\frac{m(v-u)}{u}$. 12. a. $\frac{Cb}{Kb+C}$; b. $\frac{aC}{C-Ka}$.
13. a. $\frac{sa+c}{s+1}$; b. $\frac{sb-c+b}{s}$. 14. a. $\frac{273(h-k)}{k}$; b. $\frac{273h}{273+t}$.

15. *a.* $lMt + M$; *b.* $\frac{L}{lt+1}$. **16.** *a.* 89.78^+; *b.* $\frac{w(600\,d^2 + l^2)}{3000\,d^2}$.

17. *a.* $.34_+$; *b.* $\frac{ad^2}{p-d}$. **18.** *a.* $\frac{nE}{nR+r}$; *b.* $\frac{Cr}{E-CR}$; *c.* $\frac{n(E-CR)}{C}$.

19. *a.* 585.93^+; *b.* $\frac{mv^2}{Fg}$; *c.* $\frac{Fgr}{v^2}$.

20. *a.* 120; *b.* $\frac{fg}{q-f}$; *c.* $\frac{fp}{p-f}$.

EXERCISE 94. Pages 206–208.

1. Mississippi, 3000 mi.; St. Lawrence, 2200 mi.; Yukon, 2100 mi.; Arkansas, 2000 mi.; San Francisco, 1400 mi.; Columbia, 1200 mi.

2. Corn cracked, 28 oz.; corn meal, 20 oz.; oats, 16 oz.; oats ground, 12 oz.; wheat, 30 oz.; cottonseed meal, 20 oz.

3.

AB	5.	1.	10.	40.	3.
CD	8.5	1.7	17.	68.	5.1
EF	3.	.6	6.	24.	1.8
GH	1.	.2	2.	8.	.6
ZW	5.	1.	10.	40.	3.
MN	4.	.8	8.	32.	2.4
XY	2.5	.5	5.	20.	1.5

EXERCISE 95. Pages 210–211.

2. 7:30, 16°; 8:30, 19°; 11:30, 20.5°; 2:30, 9°; 5:30, – 5°.

6. *B*, 1:12; *C*, 1:30; *D*, 1:48; *E*, 2:24; *G*, 3:30; *H*, 4:06; *I*, 4:30.

8. $33.

EXERCISE 96. Pages 212–213.

1. *A*, + 2; *B*, – 4; *C*, – 3; *D*, + 6.

2. *A*, + 5; *B*, + 2; *C*, – 2; *D*, – 5

3. *E*, (+ 5, + 2.5); *F*, (– 2.5, + 3); *G*, (– 3.5, – 4.5); *H*, (+ 4.5, – 3).

5. A parallelogram; point (+ 6, 3.5).

6. A hexagon; point (5.5, – 5).

EXERCISE 97. Page 214.

1. *B* (5, – 6); yes. *C*, (1, 2); yes. *D*, (– 2, 8); yes.

2. They do not. **3.** A straight line. **4.** They do.

5. A straight line. **6.** They do.

EXERCISE 99. Pages 219–220.

1. (7, − 2).
2. (3, 1).
3. (2, 5).
4. (− 2, 3).
5. Dependent.
6. (4, − 5.)
7. Dependent.
8. (− 2, − 3).
9. They have a common solution ; (3, 4).
10. They do not have a common solution.

EXERCISE 100. Page 220.

3. Simultaneous ; (6, − 2).
5. Simultaneous ; (− 3, 2).

EXERCISE 101. Page 224.

1. $x = 3$; $y = 2$.
2. $a = 2$; $b = -3$.
3. $s = 1$; $r = -4$.
4. $m = 2$; $n = 3$.
5. $c = 3$; $d = -5$.
6. $a = -3$; $b = 2$.
7. $x = -\frac{3}{4}$; $y = \frac{3}{4}$.
8. $s = 6$; $t = -2$.
9. $p = -1$; $q = -3$.
10. $r = -\frac{5}{2}$; $y = -\frac{3}{2}$.
11. $c = 5$; $t = -4\frac{1}{9}$.
12. $x = 3$; $y = 2$.
13. $x = -\frac{6}{5}$; $y = \frac{4}{5}$.
14. $v = \frac{1}{2}$; $g = \frac{1}{3}$.
15. $x = \frac{1}{4}$; $y = \frac{1}{6}$.
16. $x = 5$; $y = -3$.
17. $x = 4$; $y = -2$.
18. $r = 3$; $s = 11$.
19. $m = 2$; $n = -\frac{1}{2}$.
20. $a = -4$; $b = 10$.

EXERCISE 102. Page 226.

1. $m = 3$; $n = 4$.
2. $r = -4$; $s = -1$.
3. $x = 2$; $y = -1$.
4. $x = -1$; $y = 3$.
5. $x = 1$; $y = 2$.
6. $p = 5$; $q = -7$.
7. $m = -5$; $n = \frac{1}{2}$.
8. $x = 2$; $y = 6$.
9. $x = 1$; $y = \frac{1}{3}$.
10. $r = 3$; $t = -7$.
11. $p = \frac{7}{4}$; $q = \frac{8}{5}$.
12. $x = -3$; $w = -\frac{4}{3}$.
13. $A = -2$; $B = 2$.
14. $M = 5$; $N = -2$.
15. $p = \frac{1}{4}$; $q = 1$.
16. $x = \frac{3}{2}$; $y = \frac{2}{5}$.
17. $c = \frac{2}{3}$; $d = -\frac{3}{2}$.
18. $e = -3$; $f = -2$.
19. $g = -2$; $k = \frac{1}{2}$.
20. $r = 4$; $s = -5$.

EXERCISE 103. Pages 227-228.

1. $x = 6;\ y = -10.$ 2. $p = 12;\ q = -12.$
3. $m = 1;\ y = -2.$ 4. $r = 6;\ s = 2.$
5. $t = \frac{2}{3};\ w = -\frac{1}{2}.$ 6. $r = 18;\ t = 6.$
7. $c = -1;\ d = 5.$ 8. $x = 4;\ y = -3.$
9. $m = -3;\ n = 5.$ 10. $a = -8;\ b = 5.$
11. $x = -1;\ y = -5.$ 12. $x = -6;\ y = -8.$
13. $m = 3;\ n = -5.$ 14. $r = 4;\ s = -5.$
15. $p = 10;\ w = 7.$ 16. $x = 5;\ y = -4.$
17. $a = \frac{1}{2};\ b = -2.$

EXERCISE 104. Pages 229-230.

1. $a = -3;\ b = 5.$ 2. $c = 4;\ d = -6.$
3. $p = \frac{3}{4};\ q = -\frac{2}{5}.$ 4. $x = \frac{1}{3};\ y = \frac{1}{2}.$
5. $r = \frac{1}{2};\ s = -\frac{1}{2}.$ 6. $w = -5;\ v = 2.$
7. $A = \frac{2}{3};\ B = \frac{3}{4}.$ 8. $r = \frac{2}{5};\ t = -\frac{8}{7}.$
9. $x = -6;\ y = -2.$ 10. $x = 3;\ y = 4.$

EXERCISE 105. Pages 231-236.

1. 24, 35. 2. 12, 20. 3. $\frac{10}{9}$. 4. $\frac{15}{19}$.
5. Anthracite, $4.50; bituminous, $2.60. 6. A, 24 yr.; B, 40 yr.
7. 15, 26. 8. Tower, 510 ft.; statue, 37 ft.
9. $\frac{9}{16}$. 10. Length, 486 ft.; width, 470 ft.
11. Base, 40 in.; equal sides, 50 in. 12. A, 35 yr.; B, 27 yr.
13. Yale, 1700; Princeton, 1746.
14. Short side, 15 in.; long side, 25 in.
15. Oxford, 872; Harvard, 1636.
16. First weight, 15 lb.; second weight, 20 lb.
17. 7, 5. 18. $\frac{2}{3}$, $\frac{4}{5}$.
19. 13 dimes, 21 quarters. 20. A, 68 yr.; B, 18 yr.
21. Length, 21 ft.; width, 17 ft. 22. Length, 85 ft.; width, 64 ft.
23. First weight, $61\frac{1}{4}$ lb.; second weight, 95 lb.
24. John, 70 lb.; James, 80 lb.
25. Crew, 10 mi. per hour in still water; current, 2 mi. per hour

6 13 9 4a
5

26. Distance, 72 mi.; current, 3 mi. per hour.

27. Distance, 8 mi.; crew, 4 mi. per hour in still ~~water.~~

28. Slow train, 25 mi. per hour; express train, 40 mi. per hour.

29. Length, 75 ft.; width, 54 ft.

30. Fore-wheel, 9 ft.; hind-wheel, 15 ft.

31. \$1500 at 6%; \$1000 at 4½%. 32. \$2000 at 5%; \$3000 at 6%.

33. \$1200 at 4%; \$500 at 5%. 34. \$800, 6 yr.; \$300, 10 yr.

35. 105°, 75°. 36. 35°, 13.5°, 131.5°.

37. Ordinary train, 37 mi. per hour; fast train, 55.5 mi. per hour.

38. A, 14 da.; B, 10.5 da. 39. A, 12 da.; B, 18 da.

40. First man, \$15,000 at 4½%; second man, \$18,000 at 3½%; third man, \$13,000 at 5½%.

EXERCISE 106. Pages 237–238.

6. 84. 7. 72. 8. 97. 9. 83. 10. 59.

EXERCISE 107. Pages 239–241.

1. $x=\frac{35a+24b}{23}$; $y=\frac{14a-18b}{23}$.

2. $x=\frac{6a+5b}{2a+3b}$; $y=\frac{-8}{2a+3b}$. 3. $x=\frac{np-mq}{4p-m}$; $y=\frac{4q-n}{4p-m}$.

4. $x=\frac{2b+c}{5a}$; $y=\frac{b-2c}{5}$. 5. $x=\frac{dp-en}{dm-cn}$; $y=\frac{em-cp}{dm-cn}$.

6. $x=a+b$; $y=2a$. 7. $x=-2a$; $y=b$.

8. $x=\frac{3b}{2}$; $y=-\frac{a}{2}$. 9. $x=2a$; $y=a-b$.

10. $x=c-d$; $y=2c+d$. 11. $x=\frac{1}{a}$; $y=\frac{1}{b}$.

12. $x=s$; $y=-r$. 13. $x=-\frac{m+n}{m}$; $y=-\frac{m+n}{n}$.

14. $x=\frac{b-a^2}{1-ab}$; $y=\frac{a-b^2}{1-ab}$. 15. $\frac{m+n}{2}$; $\frac{m-n}{2}$.

16. Larger, $\frac{cd}{1+d}$; smaller, $\frac{c}{1+d}$. 17. Larger, $\frac{rs+t}{1+s}$; smaller, $\frac{r-t}{1+s}$.

18. A, $\frac{4k}{10-k}$; B, $\frac{4k}{k-6}$.

19. Tens' digit, $\frac{ab-a}{9+b}$; units' digit, $\frac{10a}{9+b}$.

20. $\frac{3b-4a}{2b-3a}$.

21. First, $\frac{ac+d}{a+b}$; second, $\frac{bc-d}{a+b}$.

22. Greater, $\frac{a^2-b^2}{ac-bd}$; less, $\frac{a^2-b^2}{bc-ad}$.

23. $T=\frac{mng}{m+n}$; $f=\frac{mg}{m+n}$.

24. $T=\frac{2mng}{m+n}$; $f=\frac{(m-n)g}{m+n}$.

EXERCISE 108. Pages 242–243.

1. $a=-4$; $b=-5$; $c=-6$.
2. $x=-1$; $y=-2$; $z=2$.
3. $m=2$; $n=5$; $p=-1$.
4. $r=2$; $s=-3$; $t=5$.
5. $r=4$; $t=-3$; $x=-5$.
6. $A=\frac{1}{2}$; $B=3$; $C=-2$.
7. $x=\frac{1}{2}$; $y=\frac{1}{3}$; $z=\frac{1}{4}$.
8. $x=-\frac{3}{2}$; $y=\frac{8}{13}$; $z=-\frac{3}{10}$.
9. $x=\frac{5b-a-3c}{6}$; $y=\frac{a+b-3c}{6}$; $z=\frac{a-b+c}{2}$.
10. $x=\frac{-2}{b+c}$; $y=\frac{-2}{a+c}$; $z=\frac{-2}{a+b}$.
11. A, 60°; B, 40°; C, 80°.
12. a, 80 in.; b, 50 in.; c, 45 in.
13. a, 84 in.; b, 63 in.; c, 72 in.
14. Atlantic, 35,000,000 sq. mi.; Pacific, 71,000,000 sq. mi.; Indian, 28,000,000 sq. mi.
15. Water, 1 cup; sugar, $2\frac{1}{2}$ cups; glucose, $\frac{1}{3}$ cup.
17. 896.
18. A, 15 days; B, 30 days; C, 60 days.

EXERCISE 109. Page 245.

16. ± 56.
17. $\pm 54x^2y$.
18. $\pm 65ab$.
19. $\pm 75m^2n^3$.
20. $\pm 72r^2s$.
21. $\pm 105xy^2z$.

EXERCISE 110. Pages 246–247.

4. $\pm(2x^2+x+1)$.
5. $\pm(a^2-3a+1)$.
6. $\pm(3x^2-4x-2)$.
7. $\pm(3a^2-5a+4)$.
8. $\pm(5x^2-2xy-3y^2)$.
9. $\pm(2a^3+3a^2-1)$.

10. $\pm(4m^2 + mx^2 - 3x^4)$. 11. $\pm(1 - x + x^2 - x^3)$.
12. $\pm(x^3 - 2xa^2 + 5a^3)$. 13. $\pm(3x + 5y - 4z)$.
14. $\pm(a - b - c)$. 15. $\pm(3a^2 - 2ab - 5b^2)$.
16. $\pm(2x^2 + 5x - 7)$. 17. $\pm(7m^2 - mn - 4n^2)$.
18. $\pm\left(m + 4 - \frac{2}{m}\right)$. 19. $\pm\left(x - \frac{y}{2} - \frac{3y^2}{2x}\right)$.
20. $\pm\left(\frac{4}{3} + \frac{x}{a} - \frac{2x^2}{a^2}\right)$.

EXERCISE 111. Page 250.

1. ± 39. 2. ± 67. 3. ± 75. 4. ± 94.
5. ± 153. 6. ± 234. 7. ± 417. 8. ± 205.
9. ± 21.5. 10. ± 3.14. 11. ± 3.46. 12. ± 4.23.

EXERCISE 112. Page 250.

1. ± 1.732. 2. ± 2.236. 3. ± 2.449. 4. ± 2.645.
5. ± 3.162. 6. ± 3.316. 7. ± 3.605. 8. ± 3.741.
9. ± 3.872. 10. ± 4.123. 11. ± 4.358. 12. ± 4.582.

EXERCISE 113. Page 251.

1. ± 1.000. 2. ± 1.414. 3. ± 1.732. 4. ± 2.000.
5. ± 2.236. 6. ± 2.449. 7. ± 2.645. 8. ± 2.828.
9. ± 3.000. 10. ± 3.162. 11. ± 3.316. 12. ± 3.464.
13. ± 3.605. 14. ± 3.741. 15. ± 3.872. 16. ± 4.000.
17. ± 4.123. 18. ± 4.242. 19. ± 4.358. 20. ± 4.472.
21. ± 4.582. 22. ± 4.690. 23. ± 4.795. 24. ± 4.898.
25. ± 5.000. 26. ± 5.099. 27. ± 5.196. 28. ± 5.291.
29. ± 5.385. 30. ± 5.477. 31. ± 5.567. 32. ± 5.656.
33. ± 5.744. 34. ± 5.830. 35. ± 5.916. 36. ± 6.000.
37. ± 6.082. 38. ± 6.164. 39. ± 6.245. 40. ± 6.324.
41. ± 6.403. 42. ± 6.480. 43. ± 6.557. 44. ± 6.633.
45. ± 6.708. 46. ± 6.782. 47. ± 6.855. 48. ± 6.928.
49. ± 7.000. 50. ± 7.071.

EXERCISE 114. Page 252.

1. $\pm\frac{1}{2}\sqrt{3}$, or $\pm .866$.
2. $\pm\frac{1}{3}\sqrt{5}$, or $\pm .745$.
3. $\pm\frac{1}{4}\sqrt{7}$, or $\pm .661$.
4. $\pm\frac{1}{3}\sqrt{3}$, or $\pm .577$.
5. $\pm\frac{1}{2}\sqrt{2}$, or $\pm .707$.
6. $\pm\frac{1}{5}\sqrt{10}$, or $\pm .632$.
7. $\pm\frac{1}{7}\sqrt{21}$, or $\pm .654$.
8. $\pm\frac{1}{4}\sqrt{10}$, or $\pm .790$.
9. $\pm\frac{1}{6}\sqrt{30}$, or $\pm .912$.
10. $\pm\frac{1}{7}\sqrt{28}$, or $\pm .755$.
11. $\pm\frac{1}{5}\sqrt{45}$, or ± 1.341.
12. $\pm\frac{1}{10}\sqrt{30}$, or $\pm .547$.
13. $\pm\frac{1}{11}\sqrt{77}$, or $\pm .797$.
14. $\pm\frac{1}{6}\sqrt{10}$, or $\pm .527$.
15. $\pm\frac{1}{9}\sqrt{6}$, or $\pm .272$.

EXERCISE 115. Page 253.

1. $7\sqrt{3}$, or 12.124.
2. $4\sqrt{2}$, or 5.656.
3. $2\sqrt{5}$, or 4.472.
4. $5\sqrt{3}$, or 8.660.
5. $5\sqrt{7}$, or 13.225.
6. $4\sqrt{2}$, or 5.656.
7. $\frac{7}{3}\sqrt{3}$, or 4.041.
8. $\frac{3}{10}\sqrt{10}$, or .948.
9. $\frac{1}{9}(3\sqrt{6}+\sqrt{3})$, or 1.008.
10. $-\frac{7}{4}\sqrt{6}$, or -4.285.
11. $\frac{1}{3}(2+\sqrt{5})$, or 1.412.
12. $\frac{1}{2}(1-\sqrt{3})$, or $-.366$.
13. $\frac{1}{5}(3+\sqrt{2})$, or .882.
14. $\frac{1}{7}(\sqrt{2}-1)$, or .059.
15. $-\frac{3}{11}$, or $-.272$.
16. $\frac{2}{3}(1+\sqrt{3}$, or 1.821.
17. $\frac{1}{6}(\sqrt{6}-5)$, or $-.425$.
18. $\frac{1}{7}(\sqrt{14}-3)$, or .105.
19. $\frac{1}{8}(-3+2\sqrt{10})$, or .415.
20. $\frac{1}{5}(3-\sqrt{95})$, or -1.349.

EXERCISE 116. Pages 255-256.

1. ± 5.
2. ± 4.
3. ± 6.
4. ± 2.
5. $\pm\sqrt{3}$, or ± 1.732.
6. $\pm\sqrt{15}$, or ± 3.872.
7. $\pm\frac{1}{3}\sqrt{7}$, or $\pm .881$.
8. $\pm\frac{2}{11}\sqrt{11}$, or $\pm .603$.
9. $\pm\frac{2}{3}\sqrt{3}$, or ± 1.154.
10. $\pm\sqrt{6}$, or ± 2.449.
11. ± 2.
12. $\pm\sqrt{11}$, or ± 3.316.
13. ± 8.
14. $\pm\frac{1}{2}$.
16. $\pm\sqrt{m-a^2}$.
17. $\pm\frac{1}{2}\sqrt{2(c-a)}$.
18. $\pm\frac{1}{a}\sqrt{am}$.
19. $\pm\frac{d}{2c}\sqrt{2c}$.
20. $\pm\frac{1}{h}\sqrt{2ch}$.
21. $\pm\frac{1}{3m}\sqrt{3m(n+p)}$.

EXERCISE 117. Pages 257-258.

1. 27+ ft. **2.** 37.4 ft. **3.** 21.5 in. and 64.5 in. **4.** $4.1\,a$
5. *a.* $\pm\sqrt{b^2-c^2}$; *b.* $\pm\sqrt{b^2-a^2}$. **6.** 12. **7.** 48. **8.** $2.8\,a$
9. 8.6. **10.** *a.* $.8\,s$; $.4\,s^2$ **11.** 254.5 sq. in. **12.** 117.7 ft.
13. *a.* $\frac{1}{22}\sqrt{154\,A}$; *b.* $\frac{1}{\pi}\sqrt{\pi A}$. **14.** 1021.4 cu. ft. **15.** $\frac{1}{2}\sqrt{69}$.
16. $\frac{1}{\pi h}\sqrt{\pi Vh}$. **17.** $\frac{1}{2\,\pi}\sqrt{\pi S}$. **18.** $\frac{1}{\pi h}\sqrt{3\,\pi Vh}$.
19. *a.* 144 ft.; *b.* 400 ft. **20.** 7.5 sec. **21.** 5.8 sec. **22.** $\frac{1}{g}\sqrt{2\,gs}$.
23. $\frac{1}{m}\sqrt{fmr}$.

EXERCISE 118. Page 259.

1. $8, -3$. **2.** $3, -2.5$. **3.** $-4, -2.5$. **4.** $1.25, -\frac{2}{3}$.
5. $0, 1.4$. **6.** $0, 5\frac{4}{9}$. **7.** $-a, \frac{a}{2}$. **8.** $\frac{-m}{4}, \frac{-3\,m}{2}$.
9. $\frac{5}{2}k, -\frac{2}{3}k$. **10.** $5\,p, \frac{3}{5}p$. **11.** $5, -3.5$. **12.** $-2, \frac{5}{6}$.
13. $\frac{1}{3}, \frac{3}{4}$. **14.** $-10, -1.6$. **15.** $0, -5$. **16.** $5, 5\frac{1}{3}$
17. $10, 5$. **18.** $2, 13$. **19.** $-3, 2.8$. **20.** $-7, 1.5$.

EXERCISE 119. Page 262.

1. -3. **2.** 3.5. **3.** $4, -4$. **4.** $4, -3$. **5.** $6, 1$. **6.** $-5, -1$.

EXERCISE 120. Pages 265-266.

1. $5, -1$. **2.** $3, -11$. **3.** $9, -3$. **4.** $-4, -6$. **5.** $5, -3$.
6 $2\pm 2\sqrt{2}$; or $4.828, -.828$. **7.** $-3\pm\sqrt{10}$; or $.162, -6.162$.
8 $1\pm\sqrt{2}$; or $2.414, -.414$. **9.** $4\pm 2\sqrt{5}$; or $8.472, -.472$.
10 $5\pm\sqrt{30}$; or $10.477, -.477$. **11.** $-4, -1$. **12.** $5, 2$.
13 $2, 1$. **14.** $-6, 5$. **15.** $10, 3$. **16.** $6, -1$.
17. $\frac{-9\pm 5\sqrt{5}}{2}$; or $1.090, -10.090$. **18.** $\frac{15\pm\sqrt{161}}{2}$; or $13.844, 1.156$.
20 $2, -\frac{2}{3}$. **21.** $2, -\frac{4}{5}$. **22.** $\frac{-3\pm\sqrt{58}}{7}$; or $.659, -1.516$
23. $1, -\frac{2}{3}$. **24.** $1, -\frac{2}{5}$. **25.** $\frac{5\pm\sqrt{473}}{8}$; or $3.343, -2.093$.
26. $1, -\frac{1}{3}$. **27.** $\frac{5}{3}, -1$. **28.** $\frac{3}{5}, -1$. **29.** $\frac{3}{2}, \frac{1}{2}$. **30.** $\frac{7}{3}, -1$

31. $-\frac{3}{2}, 3.$

32. $\dfrac{-3 \pm \sqrt{29}}{10}$; or $-.838, .238$

33. $\dfrac{5 \pm \sqrt{265}}{12}$; or $1.773, -.939.$

34. $-\frac{3}{5}, \frac{1}{5}.$

35. $\dfrac{1 \pm \sqrt{2}}{2}$; or $1.207, -.207.$

36. $-6 \pm \sqrt{42}$; or $.480, -12.480$

37. $-1, \frac{7}{5}.$

38. $\dfrac{1 \pm \sqrt{41}}{2}$; $3.701, -2.701.$

39. $-1, \frac{4}{5}.$

40. $\dfrac{5 \pm \sqrt{67}}{3}$; or $4.395, -1.061.$

41. $\dfrac{8 \pm \sqrt{94}}{3}$; or $5.898, -.565.$

42. $\frac{2}{3}, -\frac{6}{5}.$

43. $3 \pm \sqrt{3}$; or $4.732, 1.268.$

44. $6, 8.$

45. $\dfrac{4 \pm \sqrt{13}}{3}$; or $2.535, .131.$

EXERCISE 121. Page 267.

1. $-a \pm \sqrt{a^2 + 3}.$

2. $-a \pm \sqrt{a^2 - b}.$

3. $-2 \pm \sqrt{4 + c}.$

4. $\dfrac{-3 \pm \sqrt{9 - 4m}}{2}.$

5. $\dfrac{-3 \pm \sqrt{8n + 9}}{4}.$

6. $\dfrac{-2a \pm \sqrt{4a^2 - 2b}}{2}.$

7. $\dfrac{-3a \pm \sqrt{16t + 9a^2}}{2}.$

8. $1, -(2m + 1).$

9. $2a \pm 3b.$

10. $\dfrac{-1 \pm \sqrt{1 - a}}{a}.$

11. $\dfrac{-2t \pm \sqrt{4t^2 + 5r}}{r}.$

12. $\dfrac{-d \pm \sqrt{d^2 - cp}}{c}.$

13. $x = \dfrac{-m \pm \sqrt{m^2 - 4n}}{2}.$

14. $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$

EXERCISE 122. Page 269.

1. $1, \frac{3}{4}.$

2. $-\frac{5}{3}, -\frac{1}{2}.$

3. $12, -6.$

4. $10, -3.$

5. $\frac{11}{3}, -3.$

6. $\dfrac{-5 \pm \sqrt{13}}{6}$; or, $-1.434, -.232$

7. $\dfrac{5 \pm \sqrt{5}}{10}$; or, $.723, .276.$

8. $\dfrac{11 \pm \sqrt{73}}{12}$; or, $1.628, .204.$

9. $\dfrac{3 \pm \sqrt{17}}{4}$; or, $1.780, -.280.$

10. $\dfrac{-1 \pm \sqrt{5}}{2}$; or, $.618, -1.618.$

11. $2 \pm \sqrt{2}$; or, 3.414, .586.

12. $2, \frac{1}{8}$.

13. $\frac{2}{5}, -2$.

14. $\frac{-8 \pm \sqrt{37}}{9}$; or, $-.213$, -1.564.

15. $\frac{5}{3}, -\frac{1}{5}$.

16. $3, -\frac{3}{2}$.

17. $\frac{-1 \pm \sqrt{145}}{6}$; or, -2.173, 1.840.

18. $\frac{5 \pm \sqrt{19}}{2}$; or, 4.679, .321.

19. $\frac{2 \pm \sqrt{19}}{5}$; or, 1.271, $-.471$.

EXERCISE 123. Pages 270-272.

1. $-6, \frac{1}{4}$.

2. $-3, \frac{17}{5}$.

3. $-6, 5$.

4. $6 \pm \sqrt{3}$; or 7.732, 4.268.

5. $\frac{-5 \pm \sqrt{181}}{6}$; or, -3.075, 1.408.

6. $3, -\frac{4}{5}$.

7. $5 \pm \sqrt{30}$; or, 10.477, $-.477$.

8. $1, -\frac{2}{3}$.

9. $\frac{5 \pm 5\sqrt{5}}{2}$; or, 8.090, -3.090.

10. $\frac{1}{4}, \frac{1}{2}$.

11. $\frac{5 \pm \sqrt{29}}{2}$; or, 5.192, $-.192$.

12. $-\frac{19}{8}, 3$.

13. $-3, -4$.

14. $-1, 4$.

15. $-3, \frac{5}{7}$.

16. $2, \frac{15}{4}$.

17. -1.

18. $\frac{-11 \pm \sqrt{329}}{4}$. or, 1.784 -7.284.

19. $-\frac{1}{9}, -\frac{1}{8}$.

20. $-\frac{2}{7}$.

22. $5c, c$.

23. $\frac{r(1 \pm \sqrt{7})}{3}$.

24. $p \pm \sqrt{p^2 - c^2}$.

25. $\frac{p \pm \sqrt{p^2 - 4c^2 + 4d^2}}{2}$.

26. $\frac{-a \pm \sqrt{a^2 + 2gs}}{g}$.

27. $1, 2P - 1$.

28. $a, -1$.

29. $-a, -b$.

30 $1, -\frac{2b}{a}$.

31. $\frac{3n}{m}$; or, $-2n$.

EXERCISE 124. Pages 272-275.

1. $1, -\frac{3}{5}$. 2. $-2, \frac{5}{3}$. 3. 17, 18; $-18, -17$.
4. 4, 5, 6; $-6, -5, -4$. 5. 18, 19; $-19, -18$.
6. $\frac{2}{3}$; $\frac{3}{2}$. 7. Width, 17 ft.; length, 21 ft.
8. $\frac{8}{5}$. 9. Length, 220 ft.; width, 130 ft.
10. Altitude, 7 in.; base, 18 in. 11. Altitude, 32 ft.; base, 18 ft.
12. Altitude, 49 ft.; base, 25 ft. 13. Altitude, 125 ft.; base, 30 ft.
14. Altitude, 63 ft.; base, 27 ft. 15. Altitude, 12 ft.; base, 16 ft.
16. Altitude, 24 ft.; base, 10 ft. 17. 15 in., 15 in., 5 in.
18. 21 mi. per hour. 19. 30 mi. per hour. 20. 6 mi. per hour.
21. 20 mi. per hour. 22. Large pipe, 3 hr.; small pipe, 6 hr.
23. 3 mi. per hour. 24. 10 ft. 25. 15 rd. 26. $\frac{3}{11}$; or, $\frac{-15}{-7}$.
27. Inside city, 10 mi. per hour; outside city, 25 mi. per hour.
28. $\frac{9}{4}$. 29. Bicyclist, 8 mi. per hour; automobilist, 20 mi. per hour.
30. Fore wheel, 12 ft.; hind wheel, 16 ft.

EXERCISE 125. Page 276.

1. $\pm 7i$. 2. $\pm 6i$. 3. $\pm 10i$. 4. $\pm 9ai$.
5. $\pm 5i$. 6. $\pm 8i$. 7. $\pm 12i$. 8. $\pm 11abi$.
9. $\pm \frac{1}{2}i$. 10. $\pm \frac{2}{3}i$. 11. $\pm \frac{4}{5}i$. 12. $\pm \frac{8}{7}i$.
13. $\pm i\sqrt{3}$. 14. $\pm 4i\sqrt{2}$. 15. $\pm 3i\sqrt{2}$.
16. $\pm 6i\sqrt{2}$. 17. $\pm i\sqrt{5}$. 18. $\pm 3i\sqrt{3}$.
19. $\pm 2i\sqrt{3}$. 20. $\pm 2i\sqrt{7}$. 22. $\pm \frac{i\sqrt{2}}{3}$.
23. $\pm \frac{i\sqrt{3}}{2}$. 24. $\pm \frac{i\sqrt{5}}{6}$. 25. $\pm \frac{2i\sqrt{2}}{3}$.
26. $\pm \frac{5i\sqrt{3}}{2}$. 27. $\pm \frac{2i\sqrt{5}}{7}$. 28. $\pm \frac{4i\sqrt{2}}{5}$.
29. $\pm \frac{5i\sqrt{5}}{8}$. 30. $\pm \frac{3i\sqrt{7}}{10}$.

EXERCISE 126. Pages 277-278.

2. $9i$. 3. $-6i$. 4. i. 5. $23i$. 6. 0. 7. $7i$.
8. $2ai$. 9. $6xi$. 10. $4i\sqrt{3}$. 11. $i\sqrt{2}$. 12. $5i\sqrt{5}$.
13. $4i\sqrt{7}$. 14. $-2i\sqrt{6}$. 15. $i\sqrt{11}$. 17. $\frac{1 \pm i\sqrt{3}}{2}$

18. $\frac{2 \pm 2i}{3}$. 19. $\frac{3 \pm i\sqrt{5}}{4}$. 20. $\frac{3 \pm 2i\sqrt{7}}{5}$. 21. $\frac{1 \pm 3i\sqrt{3}}{10}$.

22. $\frac{6 \pm 5i\sqrt{3}}{13}$.

EXERCISE 127. Page 279.

1. $-1 \pm i\sqrt{2}$. 2. $\frac{-3 \pm i\sqrt{7}}{2}$. 3. $\frac{1 \pm i\sqrt{15}}{4}$.

4. $\frac{1 \pm i\sqrt{14}}{3}$. 5. $\frac{7 \pm i\sqrt{11}}{10}$. 6. $\frac{9 \pm i\sqrt{79}}{16}$.

7. $\frac{15 \pm i\sqrt{39}}{22}$. 8. $\frac{3 \pm i\sqrt{31}}{4}$. 9. $\frac{-9 \pm i\sqrt{39}}{6}$.

10. $\frac{-5 \pm i\sqrt{51}}{2}$. 11. $\frac{-5 \pm i\sqrt{29}}{9}$. 12. $\frac{(5 \pm i\sqrt{31})a}{4}$.

13. $\frac{(2 \pm i\sqrt{2})w}{8}$. 14. $\frac{(7 \pm i\sqrt{11})t}{10}$.

We can factor:

1. A polynomial whose terms hav
a common monomial factor.

$50a^2 - 20b^2 + 100c^2 = 10(5a^2 - 2b^2 + 10c^2)$

2. A perfect square trinomial

$x^2 - 6x + 9 = (x - 3)$

3. The difference of two squares.

$a^2 - b^2 = (a + b)(a + b)$

4. The trinomial by cut & try metho

$x^2 + x - 6 = (x - 2)(x + 3$

5 The difference of two cubes

$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$

6 The sum of two cubes.

$8a^6 + 125 = (2a^2 + 5)(4a^4 - 10a^2 +$

$x^2 - 4ax + 4a^2 = 9b^2 - 8a^2 + 4a^2$

$(x - 2a)^2 = 9b^2$

$x - 2a = \pm 3b$

$x = 2a \pm 3b$

Not Hep

45

13
13
39
13

14
14
96
14
186

7

$\frac{2}{5} \times \frac{10}{1} =$

18
18
144
18
324

15

15
15
315
15
225

16
16
96
16
256

4 × 4 × 4 × 4 × 4 =

64
4
+ 256

3 × 3 × 3 × 3 × 3

27
3
41
3
— 123

15
63
78

2 × 2 × 2 × 2 × 2 × 2
+ 64

4 × 4 × 4

16
4
+ 64

3 3 3 3 27
3
+ 81